Instructor's Solutions Manual

Elementary Linear Algebra with Applications

Ninth Edition

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Preface

This manual is to accompany the Ninth Edition of Bernard Kolman and David R.Hill's *Elementary Linear Algebra with Applications*. Answers to all even numbered exercises and detailed solutions to all theoretical exercises are included. It was prepared by Dennis Kletzing, Stetson University. It contains many of the solutions found in the Eighth Edition, as well as solutions to new exercises included in the Ninth Edition of the text.

Chapter 1

Linear Equations and Matrices

Section 1.1, p. 8

2. $x = 1$, $y = 2$, $z = -2$.

4. No solution.

6. $x = 13 + 10t$, $y = -8 - 8t$, t any real number.

8. Inconsistent; no solution.

10. $x = 2$, $y = -1$.

12. No solution.

14. $x = -1$, $y = 2$, $z = -2$.

16. (a) For example: $s = 0$, $t = 0$ is one answer.
    (b) For example: $s = 3$, $t = 4$ is one answer.
    (c) $s = \frac{t}{2}$.

18. Yes. The trivial solution is always a solution to a homogeneous system.

20. $x = 1$, $y = 1$, $z = 4$.

22. $r = -3$.

24. If $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$ satisfy each equation of (2) in the original order, then those same numbers satisfy each equation of (2) when the equations are listed with one of the original ones interchanged, and conversely.

25. If $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$ is a solution to (2), then the pth and qth equations are satisfied. That is,

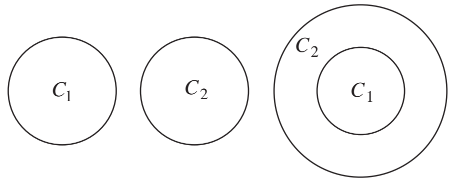
$$\begin{aligned} a_{p1}s_1 + \cdots + a_{pn}s_n &= b_p \\ a_{q1}s_1 + \cdots + a_{qn}s_n &= b_q. \end{aligned}$$

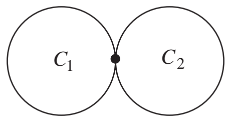
Thus, for any real number r,

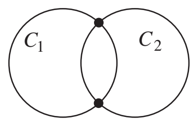
$$(a_{p1} + ra_{q1})s_1 + \cdots + (a_{pn} + ra_{qn})s_n = b_p + rb_q.$$

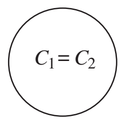
Then if the qth equation in (2) is replaced by the preceding equation, the values $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$ are a solution to the new linear system since they satisfy each of the equations.

26. (a) A unique point.
    (b) There are infinitely many points.
    (c) No points simultaneously lie in all three planes.

28. No points of intersection: 

One point of intersection: 

Two points of intersection: 

Infinitely many points of intersection: 

30. 20 tons of low-sulfur fuel, 20 tons of high-sulfur fuel.

32. 3.2 ounces of food A, 4.2 ounces of food B, and 2 ounces of food C.

34. (a)
$$\begin{aligned} p(1) &= a(1)^2 + b(1) + c = a + b + c = -5 \\ p(-1) &= a(-1)^2 + b(-1) + c = a - b + c = 1 \\ p(2) &= a(2)^2 + b(2) + c = 4a + 2b + c = 7. \end{aligned}$$
    (b) $a = 5$, $b = -3$, $c = -7$.

Section 1.2, p. 19

2. (a) $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

4. $a = 3$, $b = 1$, $c = 8$, $d = -2$.

6. (a) $C + E = E + C = \begin{bmatrix} 5 & -5 & 8 \\ 4 & 2 & 9 \\ 5 & 3 & 4 \end{bmatrix}$. (b) Impossible. (c) $\begin{bmatrix} 7 & -7 \\ 0 & 1 \end{bmatrix}$.

   (d) $\begin{bmatrix} -9 & 3 & -9 \\ -12 & -3 & -15 \\ -6 & -3 & -9 \end{bmatrix}$. (e) $\begin{bmatrix} 0 & 10 & -9 \\ 8 & -1 & -2 \\ -5 & -4 & 3 \end{bmatrix}$. (f) Impossible.

8. (a) $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$, $(A^T)^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$. (b) $\begin{bmatrix} 5 & 4 & 5 \\ -5 & 2 & 3 \\ 8 & 9 & 4 \end{bmatrix}$. (c) $\begin{bmatrix} -6 & 10 \\ 11 & 17 \end{bmatrix}$.

(d) $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$. (e) $\begin{bmatrix} 3 & 4 \\ 6 & 3 \\ 9 & 10 \end{bmatrix}$. (f) $\begin{bmatrix} 17 & 2 \\ -16 & 6 \end{bmatrix}$.

10. Yes: $2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

12. $\begin{bmatrix} \lambda - 1 & -2 & -3 \\ -6 & \lambda + 2 & -3 \\ -5 & -2 & \lambda - 4 \end{bmatrix}$.

14. Because the edges can be traversed in either direction.

16. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be an n-vector. Then

$$\mathbf{x} + \mathbf{0} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ x_2 + 0 \\ \vdots \\ x_n + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}.$$

18. $$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} &= (a_{11} + a_{12} + \cdots + a_{1m}) + (a_{21} + a_{22} + \cdots + a_{2m}) + \cdots + (a_{n1} + a_{n2} + \cdots + a_{nm}) \\ &= (a_{11} + a_{21} + \cdots + a_{n1}) + (a_{12} + a_{22} + \cdots + a_{n2}) + \cdots + (a_{1m} + a_{2m} + \cdots + a_{nm}) \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}. \end{aligned}$$

19. (a) True. $\sum_{i=1}^{n}(a_i + 1) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} 1 = \sum_{i=1}^{n} a_i + n.$

(b) True. $\sum_{i=1}^{n}\left(\sum_{j=1}^{m} 1\right) = \sum_{i=1}^{n} m = mn.$

(c) True. $$\begin{aligned} \left[\sum_{i=1}^{n} a_i\right]\left[\sum_{j=1}^{m} b_j\right] &= a_1\sum_{j=1}^{m} b_j + a_2\sum_{j=1}^{m} b_j + \cdots + a_n\sum_{j=1}^{m} b_j \\ &= (a_1 + a_2 + \cdots + a_n)\sum_{j=1}^{m} b_j \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_i b_j\right) \end{aligned}$$

20. "new salaries" $= \mathbf{u} + .08\mathbf{u} = 1.08\mathbf{u}$.

Section 1.3, p. 30

2. (a) 4. (b) 0. (c) 1. (d) 1.

4. $x = 5$.

6. $x = \pm\sqrt{2}$, $y = \pm 3$.

8. $x = \pm 5$.

10. $x = \frac{6}{5}$, $y = \frac{12}{5}$.

12. (a) Impossible. (b) $\begin{bmatrix} 0 & -1 & 1 \\ 12 & 5 & 17 \\ 19 & 0 & 22 \end{bmatrix}$. (c) $\begin{bmatrix} 15 & -7 & 14 \\ 23 & -5 & 29 \\ 13 & -1 & 17 \end{bmatrix}$. (d) $\begin{bmatrix} 8 & 8 \\ 14 & 13 \\ 13 & 9 \end{bmatrix}$. (e) Impossible.

14. (a) $\begin{bmatrix} 58 & 12 \\ 66 & 13 \end{bmatrix}$. (b) Same as (a). (c) $\begin{bmatrix} 28 & 8 & 38 \\ 34 & 4 & 41 \end{bmatrix}$.

    (d) Same as (c). (e) $\begin{bmatrix} 28 & 32 \\ 16 & 18 \end{bmatrix}$; same. (f) $\begin{bmatrix} -16 & -8 & -26 \\ -30 & 0 & -31 \end{bmatrix}$.

16. (a) 1. (b) -6. (c) $\begin{bmatrix} -3 & 0 & 1 \end{bmatrix}$. (d) $\begin{bmatrix} -1 & 4 & 2 \\ -2 & 8 & 4 \\ 3 & -12 & -6 \end{bmatrix}$. (e) 10.

    (f) $\begin{bmatrix} 9 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 1 \end{bmatrix}$. (g) Impossible.

18. $DI_2 = I_2D = D$.

20. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

22. (a) $\begin{bmatrix} 1 \\ 14 \\ 0 \\ 13 \end{bmatrix}$. (b) $\begin{bmatrix} 0 \\ 18 \\ 3 \\ 13 \end{bmatrix}$.

24. $\text{col}_1(AB) = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$; $\text{col}_2(AB) = -1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$.

26. (a) -5. (b) BA^T

28. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be $m \times p$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ be $p \times n$.

    (a) Let the ith row of A consist entirely of zeros, so that $a_{ik} = 0$ for $k = 1, 2, \ldots, p$. Then the (i, j) entry in AB is
    $$\sum_{k=1}^{p} a_{ik} b_{kj} = 0 \quad \text{for } j = 1, 2, \ldots, n.$$

    (b) Let the jth column of A consist entirely of zeros, so that $a_{kj} = 0$ for $k = 1, 2, \ldots, m$. Then the (i, j) entry in BA is
    $$\sum_{k=1}^{m} b_{ik} a_{kj} = 0 \quad \text{for } i = 1, 2, \ldots, m.$$

30. (a) $\begin{bmatrix} 2 & 3 & -3 & 1 & 1 \\ 3 & 0 & 2 & 0 & 3 \\ 2 & 3 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 2 & 3 & -3 & 1 & 1 \\ 3 & 0 & 2 & 0 & 3 \\ 2 & 3 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 5 \end{bmatrix}$.

(c) $\left[\begin{array}{ccccc|c} 2 & 3 & -3 & 1 & 1 & 7 \\ 3 & 0 & 2 & 0 & 3 & -2 \\ 2 & 3 & 0 & -4 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 5 \end{array}\right]$

32. $\begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

34. (a) $\begin{aligned} 2x_1 + x_2 + 3x_3 + 4x_4 &= 0 \\ 3x_1 - x_2 + 2x_3 &= 3 \\ -2x_1 + x_2 - 4x_3 + 3x_4 &= 2 \end{aligned}$ (b) same as (a).

36. (a) $x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$. (b) $x_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

38. (a) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

39. We have

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \mathbf{u}^T \mathbf{v}.$$

40. Possible answer: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

42. (a) Can say nothing. (b) Can say nothing.

43. (a) $\operatorname{Tr}(cA) = \sum_{i=1}^{n} c a_{ii} = c \sum_{i=1}^{n} a_{ii} = c \operatorname{Tr}(A)$.

(b) $\operatorname{Tr}(A+B) = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \operatorname{Tr}(A) + \operatorname{Tr}(B)$.

(c) Let $AB = C = \begin{bmatrix} c_{ij} \end{bmatrix}$. Then

$$\operatorname{Tr}(AB) = \operatorname{Tr}(C) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik} = \operatorname{Tr}(BA).$$

(d) Since $a_{ii}^T = a_{ii}$, $\operatorname{Tr}(A^T) = \sum_{i=1}^{n} a_{ii}^T = \sum_{i=1}^{n} a_{ii} = \operatorname{Tr}(A)$.

(e) Let $A^T A = B = \begin{bmatrix} b_{ij} \end{bmatrix}$. Then

$$b_{ii} = \sum_{j=1}^{n} a_{ij}^T a_{ji} = \sum_{j=1}^{n} a_{ji}^2 \quad \Longrightarrow \quad \operatorname{Tr}(B) = \operatorname{Tr}(A^T A) = \sum_{i=1}^{n} b_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \geq 0.$$

Hence, $\operatorname{Tr}(A^T A) \geq 0$.

44. (a) 4. (b) 1. (c) 3.

45. We have $\mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = 0$, while $\mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$.

46. (a) Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ be $m \times n$ and $n \times p$, respectively. Then $\mathbf{b}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$ and the ith entry of $A\mathbf{b}_j$ is $\displaystyle\sum_{k=1}^{n} a_{ik}b_{kj}$, which is exactly the (i, j) entry of AB.

(b) The ith row of AB is $\begin{bmatrix} \sum_k a_{ik}b_{k1} & \sum_k a_{ik}b_{k2} & \cdots & \sum_k a_{ik}b_{kn} \end{bmatrix}$. Since $\mathbf{a}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$, we have

$$\mathbf{a}_i\mathbf{b} = \begin{bmatrix} \sum_k a_{ik}b_{k1} & \sum_k a_{ik}b_{k2} & \cdots & \sum_k a_{ik}b_{kn} \end{bmatrix}.$$

This is the same as the ith row of $A\mathbf{b}$.

47. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ be $m \times n$ and $n \times p$, respectively. Then the jth column of AB is

$$\begin{aligned} (AB)_j &= \begin{bmatrix} a_{11}b_{1j} + \cdots + a_{1n}b_{nj} \\ \vdots \\ a_{m1}b_{1j} + \cdots + a_{mn}b_{nj} \end{bmatrix} \\ &= b_{1j}\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + b_{nj}\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \\ &= b_{1j}\mathrm{Col}_1(A) + \cdots + b_{nj}\mathrm{Col}_n(A). \end{aligned}$$

Thus the jth column of AB is a linear combination of the columns of A with coefficients the entries in $\mathbf{b}_j$.

48. The value of the inventory of the four types of items.

50. (a) $\mathrm{row}_1(A) \cdot \mathrm{col}_1(B) = 80(20) + 120(10) = 2800$ grams of protein consumed daily by the males.

(b) $\mathrm{row}_2(A) \cdot \mathrm{col}_2(B) = 100(20) + 200(20) = 6000$ grams of fat consumed daily by the females.

51. (a) No. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then $\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$.

(b) $\mathbf{x} = \mathbf{0}$.

52. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, and $\mathbf{c} = (c_1, c_2, \ldots, c_n)$. Then

(a) $\mathbf{a} \cdot \mathbf{b} = \displaystyle\sum_{i=1}^{n} a_ib_i$ and $\mathbf{b} \cdot \mathbf{a} = \displaystyle\sum_{i=1}^{n} b_ia_i$, so $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

(b) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \displaystyle\sum_{i=1}^{n}(a_i + b_i)c_i = \sum_{i=1}^{n} a_ic_i + \sum_{i=1}^{n} b_ic_i = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$.

(c) $(k\mathbf{a}) \cdot \mathbf{b} = \displaystyle\sum_{i=1}^{n}(ka_i)b_i = k\sum_{i=1}^{n} a_ib_i = k(\mathbf{a} \cdot \mathbf{b})$.

53. The i, ith element of the matrix AA^T is

$$\sum_{k=1}^{n} a_{ik}a_{ki}^T = \sum_{k=1}^{n} a_{ik}a_{ik} = \sum_{k=1}^{n}(a_{ik})^2.$$

Thus if $AA^T = O$, then each sum of squares $\sum_{k=1}^{n}(a_{ik})^2$ equals zero, which implies $a_{ik} = 0$ for each i and k. Thus $A = O$.

54. $AC = \begin{bmatrix} 17 & 2 & 22 \\ 18 & 3 & 23 \end{bmatrix}$. CA cannot be computed.

55. B^TB will be 6×6 while BB^T is 1×1.

Section 1.4, p. 40

1. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, $B = \begin{bmatrix} b_{ij} \end{bmatrix}$, $C = \begin{bmatrix} c_{ij} \end{bmatrix}$. Then the (i, j) entry of $A + (B + C)$ is $a_{ij} + (b_{ij} + c_{ij})$ and that of $(A + B) + C$ is $(a_{ij} + b_{ij}) + c_{ij}$. By the associative law for addition of real numbers, these two entries are equal.

2. For $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, let $B = \begin{bmatrix} -a_{ij} \end{bmatrix}$.

4. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, $B = \begin{bmatrix} b_{ij} \end{bmatrix}$, $C = \begin{bmatrix} c_{ij} \end{bmatrix}$. Then the (i, j) entry of $(A+B)C$ is $\sum_{k=1}^{n}(a_{ik} + b_{ik})c_{kj}$ and that of $AC + BC$ is $\sum_{k=1}^{n} a_{ik}c_{kj} + \sum_{k=1}^{n} b_{ik}c_{kj}$. By the distributive and additive associative laws for real numbers, these two expressions for the (i, j) entry are equal.

6. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, where $a_{ii} = k$ and $a_{ij} = 0$ if $i \neq j$, and let $B = \begin{bmatrix} b_{ij} \end{bmatrix}$. Then, if $i \neq j$, the (i, j) entry of AB is $\sum_{s=1}^{n} a_{is}b_{sj} = kb_{ij}$, while if $i = j$, the (i, i) entry of AB is $\sum_{s=1}^{n} a_{is}b_{si} = kb_{ii}$. Therefore $AB = kB$.

7. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $C = \begin{bmatrix} c_1 & c_2 & \cdots & c_m \end{bmatrix}$. Then CA is a $1 \times n$ matrix whose ith entry is $\sum_{j=1}^{n} c_j a_{ij}$.

   Since $A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, the ith entry of $\sum_{j=1}^{n} c_j A_j$ is $\sum_{j=1}^{m} c_j a_{ij}$.

8. (a) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$. (b) $\begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$. (c) $\begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$.

   (d) The result is true for $p = 2$ and 3 as shown in parts (a) and (b). Assume that it is true for $p = k$. Then

$$\begin{aligned} A^{k+1} = A^kA &= \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos k\theta\,\cos\theta - \sin k\theta\sin\theta & \cos k\theta\,\sin\theta + \sin k\theta\,\cos\theta \\ -\sin k\theta\,\cos\theta - \cos k\theta\,\sin\theta & \cos k\theta\,\cos\theta - \sin k\theta\,\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}. \end{aligned}$$

   Hence, it is true for all positive integers k.

10. Possible answers: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

12. Possible answers: $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$; $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

13. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. The (i, j) entry of $r(sA)$ is $r(sa_{ij})$, which equals $(rs)a_{ij}$ and $s(ra_{ij})$.

14. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. The (i, j) entry of $(r + s)A$ is $(r + s)a_{ij}$, which equals $ra_{ij} + sa_{ij}$, the (i, j) entry of $rA + sA$.

16. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$. Then $r(a_{ij} + b_{ij}) = ra_{ij} + rb_{ij}$.

18. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$. The (i, j) entry of $A(rB)$ is $\sum_{k=1}^{n} a_{ik}(rb_{kj})$, which equals $r \sum_{k=1}^{n} a_{ik}b_{kj}$, the (i, j) entry of $r(AB)$.

20. $\frac{1}{6}A$, $k = \frac{1}{6}$.

22. 3.

24. If $A\mathbf{x} = r\mathbf{x}$ and $\mathbf{y} = s\mathbf{x}$, then $A\mathbf{y} = A(s\mathbf{x}) = s(A\mathbf{x}) = s(r\mathbf{x}) = r(s\mathbf{x}) = r\mathbf{y}$.

26. The (i, j) entry of $(A^T)^T$ is the (j, i) entry of A^T, which is the (i, j) entry of A.

27. (b) The (i, j) entry of $(A + B)^T$ is the (j, i) entry of $\begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}$, which is to say, $a_{ji} + b_{ji}$.

    (d) Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and let $b_{ij} = a_{ji}$. Then the (i, j) entry of $(cA)^T$ is the (j, i) entry of $\begin{bmatrix} ca_{ij} \end{bmatrix}$, which is to say, cb_{ij}.

28. $(A + B)^T = \begin{bmatrix} 5 & 0 \\ 5 & 2 \\ 1 & 2 \end{bmatrix}$, $(rA)^T = \begin{bmatrix} -4 & -8 \\ -12 & -4 \\ -8 & 12 \end{bmatrix}$.

30. (a) $\begin{bmatrix} -34 \\ 17 \\ -51 \end{bmatrix}$. (b) $\begin{bmatrix} -34 \\ 17 \\ -51 \end{bmatrix}$. (c) B^TC is a real number (a 1×1 matrix).

32. Possible answers: $A = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 \\ \frac{2}{3} & 1 \end{bmatrix}$; $C = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$.

    $A = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

33. The (i, j) entry of cA is ca_{ij}, which is 0 for all i and j only if $c = 0$ or $a_{ij} = 0$ for all i and j.

34. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be such that $AB = BA$ for any 2×2 matrix B. Then in particular,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

so $b = c = 0$, $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

Also

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$
$$\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & d \\ 0 & 0 \end{bmatrix},$$

which implies that $a = d$. Thus $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for some number a.

35. We have

$$\begin{aligned} (A - B)^T &= (A + (-1)B)^T \\ &= A^T + ((-1)B)^T \\ &= A^T + (-1)B^T = A^T - B^T \quad \text{by Theorem 1.4(d)).} \end{aligned}$$

36. (a) $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
    (b) $A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{0} - \mathbf{0} = \mathbf{0}$.
    (c) $A(r\mathbf{x}_1) = r(A\mathbf{x}_1) = r\mathbf{0} = \mathbf{0}$.
    (d) $A(r\mathbf{x}_1 + s\mathbf{x}_2) = r(A\mathbf{x}_1) + s(A\mathbf{x}_2) = r\mathbf{0} + s\mathbf{0} = \mathbf{0}$.

37. We verify that $\mathbf{x}_3$ is also a solution:

$$A\mathbf{x}_3 = A(r\mathbf{x}_1 + s\mathbf{x}_2) = rA\mathbf{x}_1 + sA\mathbf{x}_2 = r\mathbf{b} + s\mathbf{b} = (r + s)\mathbf{b} = \mathbf{b}.$$

38. If $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$, then $A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

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1. (a) Let $I_m = \begin{bmatrix} d_{ij} \end{bmatrix}$ so $d_{ij} = 1$ if $i = j$ and 0 otherwise. Then the (i, j) entry of $I_m A$ is

$$\begin{aligned} \sum_{k=1}^{m} d_{ik} a_{kj} &= d_{ii} a_{ij} \quad \text{(since all other } d\text{'s} = 0) \\ &= a_{ij} \quad (\text{since } d_{ii} = 1). \end{aligned}$$

2. We prove that the product of two upper triangular matrices is upper triangular: Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ with $a_{ij} = 0$ for $i > j$; let $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ with $b_{ij} = 0$ for $i > j$. Then $AB = \begin{bmatrix} c_{ij} \end{bmatrix}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. For $i > j$, and each $1 \le k \le n$, either $i > k$ (and so $a_{ik} = 0$) or else $k \ge i > j$ (so $b_{kj} = 0$). Thus every term in the sum for c_{ij} is 0 and so $c_{ij} = 0$. Hence $\begin{bmatrix} c_{ij} \end{bmatrix}$ is upper triangular.

3. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$, where both $a_{ij} = 0$ and $b_{ij} = 0$ if $i \ne j$. Then if $AB = C = \begin{bmatrix} c_{ij} \end{bmatrix}$, we have $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = 0$ if $i \ne j$.

4. $A + B = \begin{bmatrix} 9 & -1 & 1 \\ 0 & -2 & 7 \\ 0 & 0 & 3 \end{bmatrix}$ and $AB = \begin{bmatrix} 18 & -5 & 11 \\ 0 & -8 & -7 \\ 0 & 0 & 0 \end{bmatrix}$.

5. All diagonal matrices.

6. (a) $\begin{bmatrix} 7 & -2 \\ -3 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}$ (c) $\begin{bmatrix} 20 & -20 \\ 4 & 76 \end{bmatrix}$

8. $A^pA^q = \underbrace{\underbrace{(A \cdot A \cdots A)}_{p \text{ factors}}\underbrace{(A \cdot A \cdots A)}_{q \text{ factors}}}_{p+q \text{ factors}} = A^{p+q}$; $(A^p)^q = \underbrace{A^pA^pA^p \cdots A^p}_{q \text{ factors}} = A^{\overbrace{p+p+\cdots+p}^{q \text{ summands}}} = A^{pq}$.

9. We are given that $AB = BA$. For $p = 2$, $(AB)^2 = (AB)(AB) = A(BA)B = A(AB)B = A^2B^2$. Assume that for $p = k$, $(AB)^k = A^kB^k$. Then

$$\begin{aligned}(AB)^{k+1} &= (AB)^k(AB) = A^kB^k \cdot A \cdot B = A^k(B^{k-1}AB)B \\ &= A^k(B^{k-2}AB^2)B = \cdots = A^{k+1}B^{k+1}.\end{aligned}$$

Thus the result is true for $p = k+1$. Hence it is true for all positive integers p. For $p = 0$, $(AB)^0 = I_n = A^0B^0$.

10. For $p = 0$, $(cA)^0 = I_n = 1 \cdot I_n = c^0 \cdot A^0$. For $p = 1$, $cA = cA$. Assume the result is true for $p = k$: $(cA)^k = c^kA^k$, then for $k+1$:

$$(cA)^{k+1} = (cA)^k(cA) = c^kA^k \cdot cA = c^k(A^kc)A = c^k(cA^k)A = (c^kc)(A^kA) = c^{k+1}A^{k+1}.$$

11. True for $p = 0$: $(A^T)^0 = I_n = I_n^T = (A^0)^T$. Assume true for $p = n$. Then

$$(A^T)^{n+1} = (A^T)^nA^T = (A^n)^TA^T = (AA^n)^T = (A^{n+1})^T.$$

12. True for $p = 0$: $(A^0)^{-1} = I_n^{-1} = I_n$. Assume true for $p = n$. Then

$$(A^{n+1})^{-1} = (A^nA)^{-1} = A^{-1}(A^n)^{-1} = A^{-1}(A^{-1})^n = (A^{-1})^{n+1}.$$

13. $\left(\frac{1}{k}A^{-1}\right)(kA) = \left(\frac{1}{k} \cdot k\right)A^{-1}A = I_n$ and $(kA)\left(\frac{1}{k}A^{-1}\right) = \left(k \cdot \frac{1}{k}\right)AA^{-1} = I_n$. Hence, $(kA)^{-1} = \frac{1}{k}A^{-1}$ for $k \neq 0$.

14. (a) Let $A = kI_n$. Then $A^T = (kI_n)^T = kI_n^T = kI_n = A$.
    (b) If $k = 0$, then $A = kI_n = 0I_n = O$, which is singular. If $k \neq 0$, then $A^{-1} = (kA)^{-1} = \frac{1}{k}A^{-1}$, so A is nonsingular.
    (c) No, the entries on the main diagonal do not have to be the same.

16. Possible answers: $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Infinitely many.

17. The result is false. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $AA^T = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$.

18. (a) A is symmetric if and only if $A^T = A$, or if and only if $a_{ij} = a_{ij}^T = a_{ji}$.
    (b) A is skew symmetric if and only if $A^T = -A$, or if and only if $a_{ij}^T = a_{ji} = -a_{ij}$.
    (c) $a_{ii} = -a_{ii}$, so $a_{ii} = 0$.

19. Since A is symmetric, $A^T = A$ and so $(A^T)^T = A^T$.

20. The zero matrix.

21. $(AA^T)^T = (A^T)^TA^T = AA^T$.

22. (a) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$.

(b) $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$.

23. $(A^k)^T = (A^T)^k = A^k$.

24. (a) $(A + B)^T = A^T + B^T = A + B$.

    (b) If AB is symmetric, then $(AB)^T = AB$, but $(AB)^T = B^T A^T = BA$, so $AB = BA$. Conversely, if $AB = BA$, then $(AB)^T = B^T A^T = BA = AB$, so AB is symmetric.

25. (a) Let $A = \left[a_{ij}\right]$ be upper triangular, so that $a_{ij} = 0$ for $i > j$. Since $A^T = \left[a^T_{ij}\right]$, where $a^T_{ij} = a_{ji}$, we have $a^T_{ij} = 0$ for $j > i$, or $a^T_{ij} = 0$ for $i < j$. Hence A^T is lower triangular.

    (b) Proof is similar to that for (a).

26. Skew symmetric. To show this, let A be a skew symmetric matrix. Then $A^T = -A$. Therefore $(A^T)^T = A = -A^T$. Hence A^T is skew symmetric.

27. If A is skew symmetric, $A^T = -A$. Thus $a_{ii} = -a_{ii}$, so $a_{ii} = 0$.

28. Suppose that A is skew symmetric, so $A^T = -A$. Then $(A^k)^T = (A^T)^k = (-A)^k = -A^k$ if k is a positive odd integer, so A^k is skew symmetric.

29. Let $S = \left(\frac{1}{2}\right)(A + A^T)$ and $K = \left(\frac{1}{2}\right)(A - A^T)$. Then S is symmetric and K is skew symmetric, by Exercise 18. Thus
$$S + K = \left(\tfrac{1}{2}\right)(A + A^T + A - A^T) = \left(\tfrac{1}{2}\right)(2A) = A.$$
Conversely, suppose $A = S + K$ is any decomposition of A into the sum of a symmetric and skew symmetric matrix. Then
$$\begin{aligned} A^T &= (S + K)^T = S^T + K^T = S - K \\ A + A^T &= (S + K) + (S - K) = 2S, \quad S = \left(\tfrac{1}{2}\right)(A + A^T), \\ A - A^T &= (S + K) - (S - K) = 2K, \quad K = \left(\tfrac{1}{2}\right)(A - A^T) \end{aligned}$$

30. $S = \dfrac{1}{2}\begin{bmatrix} 2 & 7 & 3 \\ 7 & 12 & 3 \\ 3 & 3 & 6 \end{bmatrix}$ and $K = \dfrac{1}{2}\begin{bmatrix} 0 & -1 & -7 \\ 1 & 0 & 1 \\ 7 & -1 & 0 \end{bmatrix}$.

31. Form $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since the linear systems
$$\begin{aligned} 2w + 3y &= 1 \\ 4w + 6y &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x + 3z &= 0 \\ 4x + 6z &= 1 \end{aligned}$$
have no solutions, we conclude that the given matrix is singular.

32. $D^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$.

34. $A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 \end{bmatrix}$.

36. (a) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 22 \end{bmatrix}$. (b) $\begin{bmatrix} 38 \\ 53 \end{bmatrix}$.

38. $\begin{bmatrix} -9 \\ -6 \end{bmatrix}$.

40. $\begin{bmatrix} 8 \\ 9 \end{bmatrix}$.

42. Possible answer: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

43. Possible answer: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & 8 \end{bmatrix}$.

44. The conclusion of the corollary is true for $r = 2$, by Theorem 1.6. Suppose $r \geq 3$ and that the conclusion is true for a sequence of $r - 1$ matrices. Then

$$(A_1 A_2 \cdots A_r)^{-1} = [(A_1 A_2 \cdots A_{r-1}) A_r]^{-1} = A_r^{-1} (A_1 A_2 \cdots A_{r-1})^{-1} = A_r^{-1} A_{r-1}^{-1} \cdots A_2^{-1} A_1^{-1}.$$

45. We have $A^{-1}A = I_n = AA^{-1}$ and since inverses are unique, we conclude that $(A^{-1})^{-1} = A$.

46. Assume that A is nonsingular, so that there exists an $n \times n$ matrix B such that $AB = I_n$. Exercise 28 in Section 1.3 implies that AB has a row consisting entirely of zeros. Hence, we cannot have $AB = I_n$.

47. Let

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{bmatrix},$$

where $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$. Then

$$A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$$

as can be verified by computing $AA^{-1} = A^{-1}A = I_n$.

48. $A^4 = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 625 \end{bmatrix}$.

49. $A^p = \begin{bmatrix} a_{11}^p & 0 & 0 & \cdots & 0 \\ 0 & a_{22}^p & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & \cdots & a_{nn}^p \end{bmatrix}$.

50. Multiply both sides of the equation by A^{-1}.

51. Multiply both sides by A^{-1}.

52. Form $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This leads to the linear systems

$$\begin{aligned} aw + by &= 1 \\ cw + dy &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} ax + bz &= 0 \\ cx + dz &= 1. \end{aligned}$$

A solution to these systems exists only if $ad - bc \neq 0$. Conversely, if $ad - bc \neq 0$ then a solution to these linear systems exists and we find A^{-1}.

53. $A\mathbf{x} = \mathbf{0}$ implies that $A^{-1}(A\mathbf{x}) = A\mathbf{0} = \mathbf{0}$, so $\mathbf{x} = \mathbf{0}$.

54. We must show that $(A^{-1})^T = A^{-1}$. First, $AA^{-1} = I_n$ implies that $(AA^{-1})^T = I_n^T = I_n$. Now $(AA^{-1})^T = (A^{-1})^T A^T = (A^{-1})^T A$, which means that $(A^{-1})^T = A^{-1}$.

55. $A + B = \left[\begin{array}{cc|c} 4 & 5 & 0 \\ 0 & 4 & 1 \\ \hline 6 & -2 & 6 \end{array}\right]$ is one possible answer.

56. $A = \left[\begin{array}{c|c|c} 2\times 2 & 2\times 2 & 2\times 1 \\ \hline 2\times 2 & 2\times 2 & 2\times 1 \\ \hline 2\times 2 & 2\times 2 & 2\times 1 \end{array}\right]$ and $B = \left[\begin{array}{c|c} 2\times 2 & 2\times 3 \\ \hline 2\times 2 & 2\times 3 \\ \hline 1\times 2 & 1\times 3 \end{array}\right]$.

$A = \left[\begin{array}{c|c} 3\times 3 & 3\times 2 \\ \hline 3\times 3 & 3\times 2 \end{array}\right]$ and $B = \left[\begin{array}{c|c} 3\times 3 & 3\times 2 \\ \hline 2\times 3 & 2\times 2 \end{array}\right]$.

$$AB = \begin{bmatrix} 21 & 48 & 41 & 48 & 40 \\ 18 & 26 & 34 & 33 & 5 \\ 24 & 26 & 42 & 47 & 16 \\ 28 & 38 & 54 & 70 & 35 \\ 33 & 33 & 56 & 74 & 42 \\ 34 & 37 & 58 & 79 & 54 \end{bmatrix}.$$

57. A symmetric matrix. To show this, let $A_1, \ldots, A_n$ be symmetric matrices and let $x_1, \ldots, x_n$ be scalars. Then $A_1^T = A_1, \ldots, A_n^T = A_n$. Therefore

$$\begin{aligned} (x_1A_1 + \cdots + x_nA_n)^T &= (x_1A_1)^T + \cdots + (x_nA_n)^T \\ &= x_1A_1^T + \cdots + x_nA_n^T \\ &= x_1A_1 + \cdots + x_nA_n. \end{aligned}$$

Hence the linear combination $x_1A_1 + \cdots + x_nA_n$ is symmetric.

58. A scalar matrix. To show this, let $A_1, \ldots, A_n$ be scalar matrices and let $x_1, \ldots, x_n$ be scalars. Then $A_i = c_iI_n$ for scalars $c_1, \ldots, c_n$. Therefore

$$x_1A_1 + \cdots + x_nA_n = x_1(c_1I_1) + \cdots + x_n(c_nI_n) = (x_1c_1 + \cdots + x_nc_n)I_n$$

which is the scalar matrix whose diagonal entries are all equal to $x_1c_1 + \cdots + x_nc_n$.

59. (a) $\mathbf{w}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 19 \\ 5 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} 65 \\ 19 \end{bmatrix}$, $\mathbf{w}_4 = \begin{bmatrix} 214 \\ 65 \end{bmatrix}$; $u_2 = 5$, $u_3 = 19$, $u_4 = 65$, $u_5 = 214$.

    (b) $\mathbf{w}_{n-1} = A^{n-1}\mathbf{w}_0$.

60. (a) $\mathbf{w}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} 16 \\ 8 \end{bmatrix}$.

    (b) $\mathbf{w}_{n-1} = A^{n-1}\mathbf{w}_0$.

63. (b) In MATLAB the following message is displayed.

```
Warning: Matrix is close to singular or badly scaled.
Results may be inaccurate.
RCOND = 2.937385e-018
```

Then a computed inverse is shown which is useless. (RCOND above is an estimate of the condition number of the matrix.)

(c) In MATLAB a message similar to that in (b) is displayed.

64. (c) In MATLAB, $AB - BA$ is not O. It is a matrix each of whose entries has absolute value less than 1×10^{-14}.

65. (b) Let $\mathbf{x}$ be the solution from the linear system solver in MATLAB and $\mathbf{y} = A^{-1}B$. A crude measure of difference in the two approaches is to look at $\max\{|x_i - y_i| \;\; i = 1, \ldots, 10\}$. This value is approximately 6×10^{-5}. Hence, computationally the methods are not identical.

66. The student should observe that the "diagonal" of ones marches toward the upper right corner and eventually "exits" the matrix leaving all of the entries zero.

67. (a) As $k \to \infty$, the entries in $A^k \to 0$, so $A^k \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) As $k \to \infty$, some of the entries in A^k do not approach 0, so A^k does not approach any matrix.

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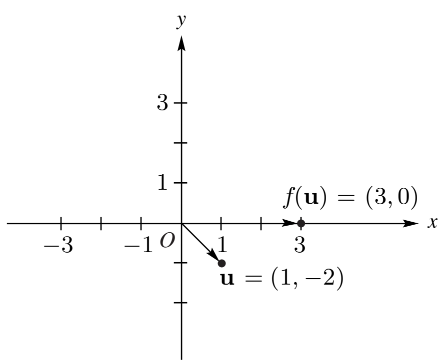
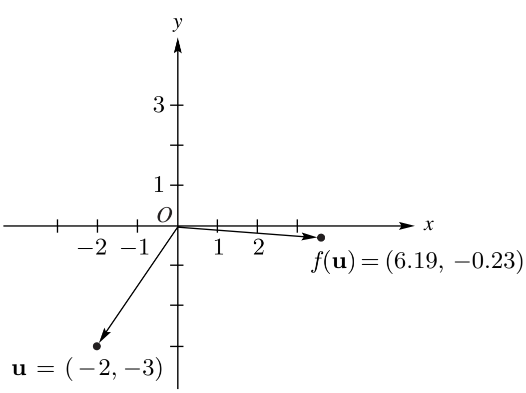
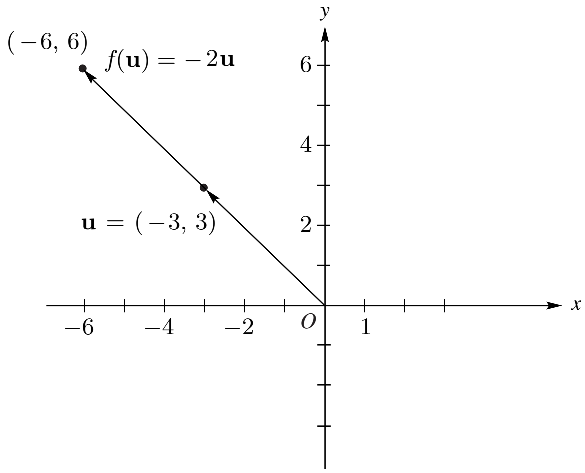
2.



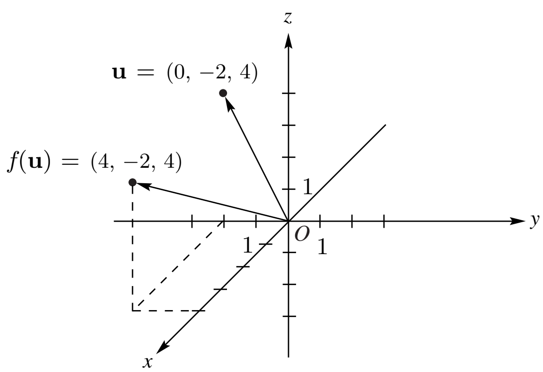
4.



6.



8.



10. No.

12. Yes.

14. No.

16. (a) Reflection about the line $y = x$.
    (b) Reflection about the line $y = -x$.

18. (a) Possible answers: $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
    (b) Possible answers: $\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

20. (a) $f(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = f(\mathbf{u}) + f(\mathbf{v})$.
    (b) $f(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cf(\mathbf{u})$.
    (c) $f(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u}) + A(c\mathbf{v}) = c(A\mathbf{u}) + d(A\mathbf{v}) = cf(\mathbf{u}) + df(\mathbf{v})$.

21. For any real numbers c and d, we have

$$f(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u}) + A(d\mathbf{v}) = c(A\mathbf{u}) + d(A\mathbf{v}) = cf(\mathbf{u}) + df(\mathbf{v}) = c\mathbf{0} + d\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

22. (a) $O(\mathbf{u}) = \begin{bmatrix} 0 & \cdots & 0 \\ & \vdots & \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}.$

(b) $I(\mathbf{u}) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}.$

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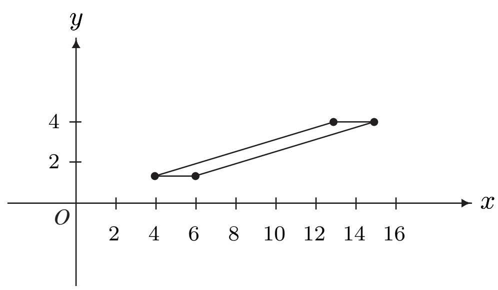
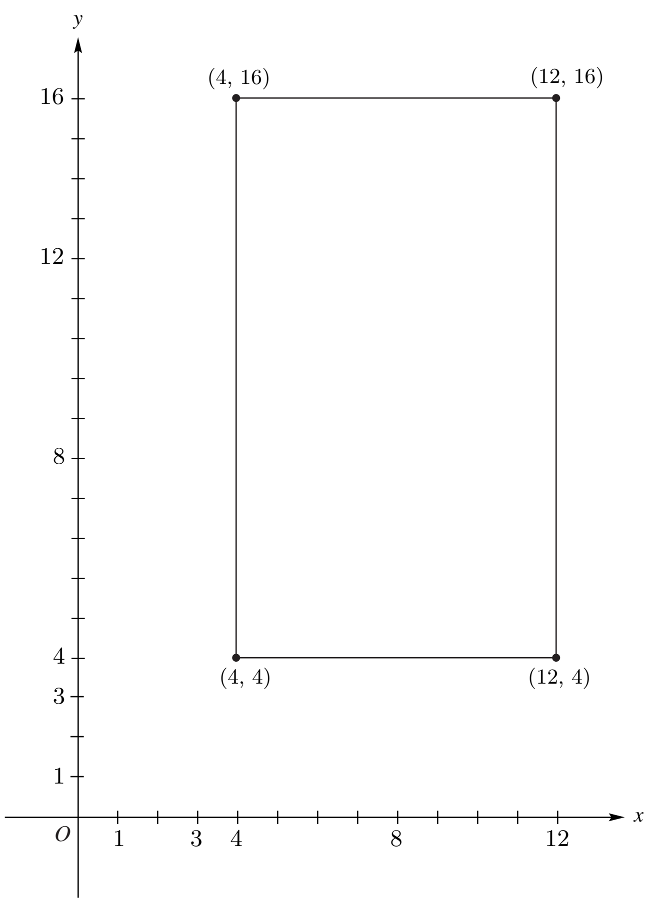
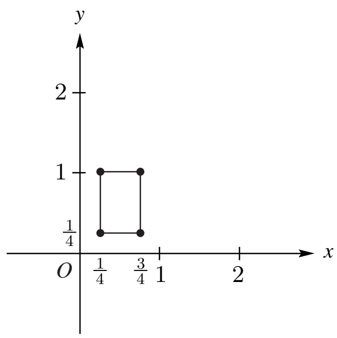
2.



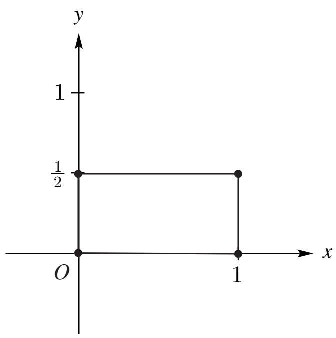
4. (a)



(b)



6.



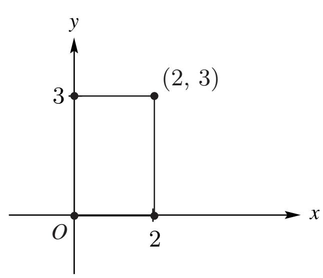
8. $(1, -2)$, $(-3, 6)$, $(11, -10)$.

10. We find that

$$\begin{aligned}(f_1 \circ f_2)(\mathbf{e}_1) &= \mathbf{e}_2\\ (f_2 \circ f_1)(\mathbf{e}_1) &= -\mathbf{e}_2.\end{aligned}$$

Therefore $f_1 \circ f_2 \neq f_2 \circ f_1$.

12. Here $f(\mathbf{u}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{u}$. The new vertices are $(0, 0)$, $(2, 0)$, $(2, 3)$, and $(0, 3)$.



14. (a) Possible answer: First perform f_1 (45° counterclockwise rotation), then f_2.
    (b) Possible answer: First perform f_3, then f_2.

16. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Then A represents a rotation through the angle θ. Hence A^2 represents a rotation through the angle 2θ, so

$$A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

Since

$$A^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix},$$

we conclude that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta.$$

17. Let

$$A = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos(-\theta_2) & -\sin(-\theta_2) \\ \sin(-\theta_2) & \cos(-\theta_2) \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}.$$

Then A and B represent rotations through the angles θ_1 and $-\theta_2$, respectively. Hence BA represents a rotation through the angle $\theta_1 - \theta_2$. Then

$$BA = \begin{bmatrix} \cos(\theta_1-\theta_2) & -\sin(\theta_1-\theta_2) \\ \sin(\theta_1-\theta_2) & \cos(\theta_1-\theta_2) \end{bmatrix}.$$

Since

$$BA = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 \end{bmatrix},$$

we conclude that

$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$$
$$\sin(\theta_1-\theta_2) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2.$$

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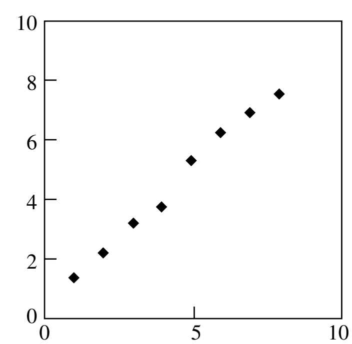
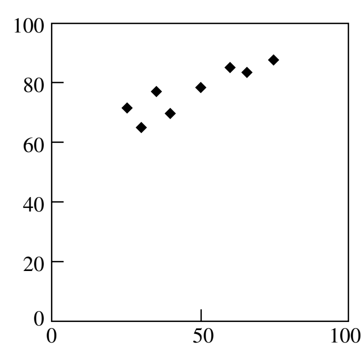
2. Correlation coefficient $= 0.9981$. Quite highly correlated.



4. Correlation coefficient $= 0.8774$. Moderately positively correlated.



Supplementary Exercises for Chapter 1, p. 80

2. (a) $k = 1 \qquad B = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$.

$k = 2 \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \end{bmatrix}$.

$k = 3 \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 \end{bmatrix}$.

$k = 4 \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) The answers are not unique. The only requirement is that row 2 of B have all zero entries.

4. (a) $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. (c) I_4.

(d) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = B$ implies

$$b(a + d) = 1$$
$$c(a + d) = 0.$$

It follows that $a + d \neq 0$ and $c = 0$. Thus

$$A^2 = \begin{bmatrix} a^2 & b(a + d) \\ b & d^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence, $a = d = 0$, which is a contradiction; thus, B has no square root.

5. (a) $(A^T A)_{ii} = (\text{row}_i A^T) \times (\text{col}_i A) = (\text{col}_i A)^T \times (\text{col}_i A)$

(b) From part (a)

$$(A^T A)_{ii} = \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ni} \end{bmatrix} \times \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} = \sum_{j=1}^{n} a_{ji}^2 \geq 0.$$

(c) $A^T A = O_n$ if and only if $(A^T A)_{ii} = 0$ for $i = 1, \dots, n$. But this is possible if and only if $a_{ij} = 0$ for $i = 1, \dots, n$ and $j = 1, \dots, n$

6. $(A^k)^T = \underbrace{(A \cdot A \cdots A)^T}_{k \text{ times}} = \underbrace{A^T A^T \cdots A^T}_{k \text{ times}} = (A^T)^k$.

7. Let A be a symmetric upper (lower) triangular matrix. Then $a_{ij} = a_{ji}$ and $a_{ij} = 0$ for $j > i$ ($j < i$). Thus, $a_{ij} = 0$ whenever $i \neq j$, so A is diagonal.

8. If A is skew symmetric then $A^T = -A$. Note that $\mathbf{x}^T A\mathbf{x}$ is a scalar, thus $(\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A\mathbf{x}$. That is,

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A^T \mathbf{x} = -(\mathbf{x}^T A\mathbf{x}).$$

The only scalar equal to its negative is zero. Hence $\mathbf{x}^T A\mathbf{x} = 0$ for all $\mathbf{x}$.

9. We are asked to prove an "if and only if" statement. Hence two things must be proved.

(a) If A is nonsingular, then $a_{ii} \neq 0$ for $i = 1, \dots, n$.
Proof: If A is nonsingular then A is row equivalent to I_n. Since A is upper triangular, this can occur only if we can multiply row i by $1/a_{ii}$ for each i. Hence $a_{ii} \neq 0$ for $i = 1, \dots, n$. (Other row operations will then be needed to get I_n.)

(b) If $a_{ii} \neq 0$ for $i = 1, \ldots, n$ then A is nonsingular.
Proof: Just reverse the steps given above in part (a).

10. Let $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$. Then A and B are skew symmetric and $AB = \begin{bmatrix} -ab & 0 \\ 0 & -ab \end{bmatrix}$ which is diagonal. The result is not true for $n > 2$. For example, let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}.$$

Then $A^2 = \begin{bmatrix} 5 & 6 & -3 \\ 6 & 10 & 2 \\ -3 & 2 & 13 \end{bmatrix}$.

11. Using the definition of trace and Exercise 5(a), we find that

$$\begin{aligned} \mathrm{Tr}(A^TA) &= \text{sum of the diagonal entries of } A^TA && \text{(definition of trace)} \\ &= \sum_{i=1}^{n} (A^TA)_{ii} = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} a_{ji}^2 \right] && \text{(Exercise 5(a))} \\ &= \text{sum of the squares of all entries of } A \end{aligned}$$

Thus the only way $\mathrm{Tr}(A^TA) = 0$ is if $a_{ij} = 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. That is, if $A = O$.

12. When $AB = BA$.

13. Let $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 1 & \frac{1}{2} + \left(\frac{1}{2}\right)^2 \\ 0 & \left(\frac{1}{2}\right)^2 \end{bmatrix} \quad \text{and} \quad A^3 = \begin{bmatrix} 1 & \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 \\ 0 & \left(\frac{1}{2}\right)^3 \end{bmatrix}.$$

Following the pattern for the elements we have

$$A^n = \begin{bmatrix} 1 & \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n \\ 0 & \left(\frac{1}{2}\right)^n \end{bmatrix}.$$

A formal proof by induction can be given.

14. $B^k = PA^kP^{-1}$.

15. Since A is skew symmetric, $A^T = -A$. Therefore,

$$A[-(A^{-1})^T] = -A(A^{-1})^T = A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and similarly, $[-(A^{-1})^T]A = I$. Hence $-(A^{-1})^T = A^{-1}$, so $(A^{-1})^T = -A^{-1}$, and therefore A^{-1} is skew symmetric.

16. If $A\mathbf{x} = \mathbf{0}$ for all $n \times 1$ matrices $\mathbf{x}$, then $AE_j = \mathbf{0}$, $j = 1, 2, \ldots, n$ where $E_j =$ column j of I_n. But then

$$AE_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = \mathbf{0}.$$

Hence column j of $A = \mathbf{0}$ for each j and it follows that $A = O$.

17. If $A\mathbf{x} = \mathbf{x}$ for all $n \times 1$ matrices X, then $AE_j = E_j$, where E_j is column j of I_n. Since

$$AE_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = E_j$$

it follows that $a_{ij} = 1$ if $i = j$ and 0 otherwise. Hence $A = I_n$.

18. If $A\mathbf{x} = B\mathbf{x}$ for all $n \times 1$ matrices $\mathbf{x}$, then $AE_j = BE_j$, $j = 1, 2, \ldots, n$ where E_j = column j of I_n. But then

$$AE_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = BE_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

Hence column j of A = column j of B for each j and it follows that $A = B$.

19. (a) $I_n^2 = I_n$ and $O^2 = O$

(b) One such matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and another is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) If $A^2 = A$ and A^{-1} exists, then $A^{-1}(A^2) = A^{-1}A$ which simplifies to give $A = I_n$.

20. We have $A^2 = A$ and $B^2 = B$.

(a) $(AB)^2 = ABAB = A(BA)B = A(AB)B \quad$ (since $AB = BA$)

$= A^2B^2 = AB \quad$ (since A and B are idempotent)

(b) $(A^T)^2 = A^TA^T = (AA)^T \quad$ (by the properties of the transpose)

$= (A^2)^T = A^T \quad$ (since A is idempotent)

(c) If A and B are $n \times n$ and idempotent, then $A + B$ need not be idempotent. For example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Both A and B are idempotent and $C = A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. However, $C^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \neq C$.

(d) $k = 0$ and $k = 1$.

21. (a) We prove this statement using induction. The result is true for $n = 1$. Assume it is true for $n = k$ so that $A^k = A$. Then

$$A^{k+1} = AA^k = AA = A^2 = A.$$

Thus the result is true for $n = k + 1$. It follows by induction that $A^n = A$ for all integers $n \geq 1$.

(b) $(I_n - A)^2 = I_n^2 - 2A + A^2 = I_n - 2A + A = I_n - A$.

22. (a) If A were nonsingular then products of A with itself must also be nonsingular, but A^k is singular since it is the zero matrix. Thus A must be singular.

(b) $A^3 = O$.

(c) $k = 1 \quad A = O;\ \ I_n - A = I_n;\ (I_n - A)^{-1}A = I_n$

$k = 2 \quad A^2 = O;\ \ (I_n - A)(I_n + A) = I_n - A^2 = I_n;\ (I_n - A)^{-1} = I_n + A$

$k = 3 \quad A^3 = O;\ \ (I_n - A)(I_n + A + A^2) = I_n - A^3 = I_n;\ (I_n - A)^{-1} = I_n + A + A^2$

etc.

24. $$\frac{\mathbf{v}\cdot\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}}$$

25. (a) $\mathrm{Mcd}(cA) = \sum_{i+j=n+1}(ca_{ij}) = c\sum_{i+j=n+1}a_{ij} = c\,\mathrm{Mcd}(A)$

(b) $\mathrm{Mcd}(A+B) = \sum_{i+j=n+1}(a_{ij}+b_{ij}) = \sum_{i+j=n+1}a_{ij} + \sum_{i+j=n+1}b_{ij} = \mathrm{Mcd}(A)+\mathrm{Mcd}(B)$

(c) $\mathrm{Mcd}(A^T) = (A^T)_{1n} + (A^T)_{2\,n-1} + \cdots + (A^T)_{n1} = a_{n1} + a_{n-1\,2} + \cdots + a_{1n} = \mathrm{Mcd}(A)$

(d) Let $A = \begin{bmatrix}7 & -3\\0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$. Then

$$AB = \begin{bmatrix}10 & 4\\0 & 0\end{bmatrix} \quad \text{with } \mathrm{Mcd}(AB) = 4$$

and

$$BA = \begin{bmatrix}7 & -3\\-7 & 3\end{bmatrix} \quad \text{with } \mathrm{Mcd}(BA) = -10.$$

26. (a) $\left[\begin{array}{cc|cc}1 & 2 & 0 & 0\\3 & 4 & 0 & 0\\\hline 0 & 0 & 1 & 0\\0 & 0 & 2 & 3\end{array}\right]$.

(b) Solve $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\mathbf{y} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}\mathbf{z} = \begin{bmatrix}0\\3\end{bmatrix}$ obtaining $\mathbf{y} = \begin{bmatrix}-1\\1\end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix}0\\1\end{bmatrix}$. Then the solution to the given linear system $A\mathbf{x} = B$ is $\mathbf{x} = \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}$ where $\mathbf{x} = \begin{bmatrix}\mathbf{y}\\\mathbf{z}\end{bmatrix}$.

27. Let

$$A = \begin{bmatrix}0 & a\\-a & 0\end{bmatrix} \quad\text{and}\quad B = \begin{bmatrix}0 & b\\-b & 0\end{bmatrix}.$$

Then A and B are skew symmetric and

$$AB = \begin{bmatrix}-ab & 0\\0 & -ab\end{bmatrix}$$

which is diagonal. The result is not true for $n > 2$. For example, let

$$A = \begin{bmatrix}0 & 1 & 2\\-1 & 0 & 3\\-2 & -3 & 0\end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 5 & 6 & -3 \\ 6 & 10 & 2 \\ -3 & 2 & 13 \end{bmatrix}.$$

28. Consider the linear system $A\mathbf{x} = \mathbf{0}$. If A_{11} and A_{22} are nonsingular, then the matrix

$$\begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}$$

is the inverse of A (verify by block multiplying). Thus A is nonsingular.

29. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$$

where A_{11} is $r \times r$ and A_{22} is $s \times s$. Let

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is $r \times r$ and B_{22} is $s \times s$. Then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_r & O \\ O & I_s \end{bmatrix}.$$

We have $A_{22}B_{22} = I_s$, so $B_{22} = A_{22}^{-1}$. We also have $A_{22}B_{21} = O$, and multiplying both sides of this equation by A_{22}^{-1}, we find that $B_{21} = O$. Thus $A_{11}B_{11} = I_r$, so $B_{11} = A_{11}^{-1}$. Next, since

$$A_{11}B_{12} + A_{12}B_{22} = O$$

then

$$A_{11}B_{12} = -A_{12}B_{22} = -A_{12}A_{22}^{-1}$$

Hence,

$$B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}.$$

Since we have solved for B_{11}, B_{12}, B_{21}, and B_{22}, we conclude that A is nonsingular. Moreover,

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{bmatrix}.$$

30. (a) $XY^T = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$. (b) $XY^T = \begin{bmatrix} -1 & 0 & 3 & 5 \\ -2 & 0 & 6 & 10 \\ -1 & 0 & 3 & 5 \\ -2 & 0 & 6 & 10 \end{bmatrix}$.

31. Let $X = \begin{bmatrix} 1 & 5 \end{bmatrix}^T$ and $Y = \begin{bmatrix} 4 & -3 \end{bmatrix}^T$. Then

$$XY^T = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \begin{bmatrix} 4 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 20 & -15 \end{bmatrix} \quad \text{and} \quad YX^T = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \begin{bmatrix} 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 20 \\ -3 & -15 \end{bmatrix}.$$

It follows that XY^T is not necessarily the same as YX^T.

32. $\operatorname{Tr}(XY^T) = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ (See Exercise 27)
$$= X^TY.$$

33. $$\operatorname{col}_1(A) \times \operatorname{row}_1(B) + \operatorname{col}_2(A) \times \operatorname{row}_2(B) = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} \begin{bmatrix} 6 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 4 \\ 6 & 12 \\ 10 & 20 \end{bmatrix} + \begin{bmatrix} 42 & 56 \\ 54 & 72 \\ 66 & 88 \end{bmatrix} = \begin{bmatrix} 44 & 60 \\ 60 & 84 \\ 76 & 108 \end{bmatrix} = AB.$$

34. (a) $H^T = (I_n - 2WW^T)^T = I_n^T - 2(WW^T)^T = I_n - 2(W^T)^TW^T = I_n - 2WW^T = H.$

(b) $$\begin{aligned} HH^T = HH &= (I_n - 2WW^T)(I_n - 2WW^T) \\ &= I_n - 4WW^T + 4WW^TWW^T \\ &= I_n - 4WW^T + 4W(W^TW)W^T \\ &= I_n - 4WW^T + 4W(I_n)W^T = I_n \end{aligned}$$
Thus, $H^T = H^{-1}$.

35. (a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 5 & -1 \\ -1 & 1 & 2 & 5 \\ 5 & -1 & 1 & 2 \\ 2 & 5 & -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_5$ (d) $\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}$

36. We have $C = \operatorname{circ}(c_1, c_2, c_3) = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{bmatrix}$. Thus C is symmetric if and only if $c_2 = c_3$.

37. $C\mathbf{x} = \left[\sum_{i=1}^{n} c_i\right]\mathbf{x}.$

38. We proceed directly.

$$C^TC = \begin{bmatrix} c_1 & c_3 & c_2 \\ c_2 & c_1 & c_3 \\ c_3 & c_2 & c_1 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{bmatrix} = \begin{bmatrix} c_1^2 + c_3^2 + c_2^2 & c_1c_2 + c_3c_1 + c_2c_3 & c_1c_3 + c_3c_2 + c_2c_1 \\ c_2c_1 + c_1c_3 + c_3c_2 & c_2^2 + c_1^2 + c_3^2 & c_2c_3 + c_1c_2 + c_3c_1 \\ c_3c_1 + c_2c_3 + c_1c_2 & c_3c_2 + c_2c_1 + c_1c_3 & c_3^2 + c_2^2 + c_1^2 \end{bmatrix}$$
$$CC^T = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{bmatrix} \begin{bmatrix} c_1 & c_3 & c_2 \\ c_2 & c_1 & c_3 \\ c_3 & c_2 & c_1 \end{bmatrix} = \begin{bmatrix} c_1^2 + c_2^2 + c_3^2 & c_1c_3 + c_2c_1 + c_3c_2 & c_1c_2 + c_2c_3 + c_3c_1 \\ c_3c_1 + c_1c_2 + c_2c_3 & c_3^2 + c_1^2 + c_2^2 & c_3c_2 + c_1c_3 + c_2c_1 \\ c_2c_1 + c_3c_2 + c_1c_3 & c_2c_3 + c_3c_1 + c_1c_2 & c_2^2 + c_3^2 + c_1^2 \end{bmatrix}.$$

It follows that $C^TC = CC^T$.

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True or False

1. False.	2. False.	3. True.	4. True.	5. True.
6. True.	7. True.	8. True.	9. True.	10. True.

Quiz

1. $\mathbf{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

2. $r = 0$.

3. $a = b = 4$.

4. (a) $a = 2$.

   (b) $b = 10$, $c =$ any real number.

5. $\mathbf{u} = \begin{bmatrix} 3 \\ r \end{bmatrix}$, where r is any real number.

Chapter 2

Solving Linear Systems

Section 2.1, p. 94

2. (a) Possible answer: $\begin{array}{l} -1\mathbf{r}_1 \to \mathbf{r}_1 \\ 3\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -4\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3 \\ 2\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3 \end{array}$ $\begin{bmatrix} 1 & -1 & 1 & 0 & -3 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   (b) Possible answer: $\begin{array}{l} 2\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -4\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3 \\ \mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3 \\ \frac{1}{6}\mathbf{r}_3 \to \mathbf{r}_3 \end{array}$ $\begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

4. (a) $\begin{array}{l} 3\mathbf{r}_3 + \mathbf{r}_1 \to \mathbf{r}_1 \\ -\mathbf{r}_3 + \mathbf{r}_2 \to \mathbf{r}_2 \end{array}$ $\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $-3\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1$ $\begin{bmatrix} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$

6. (a) $\begin{array}{l} -\mathbf{r}_1 \to \mathbf{r}_1 \\ -2\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -2\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3 \\ \frac{1}{2}\mathbf{r}_2 \to \mathbf{r}_2 \\ -3\mathbf{r}_3 \to \mathbf{r}_3 \\ \frac{4}{3}\mathbf{r}_3 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -5\mathbf{r}_3 + \mathbf{r}_1 \to \mathbf{r}_1 \\ 2\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1 \end{array}$ I_3 (b) $\begin{array}{l} -3\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -5\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3 \\ 2\mathbf{r}_1 + \mathbf{r}_4 \to \mathbf{r}_4 \\ -\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3 \\ -\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1 \end{array}$ $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. (a) REF (b) RREF (c) N

9. Consider the columns of A which contain leading entries of nonzero rows of A. If this set of columns is the entire set of n columns, then $A = I_n$. Otherwise there are fewer than n leading entries, and hence fewer than n nonzero rows of A.

10. (a) A is row equivalent to itself: the sequence of operations is the empty sequence.

    (b) Each elementary row operation of types I, II or III has a corresponding inverse operation of the same type which "undoes" the effect of the original operation. For example, the inverse of the operation "add d times row r of A to row s of A" is "subtract d times row r of A from row s of A." Since B is assumed row equivalent to A, there is a sequence of elementary row operations which gets from A to B. Take those operations in the reverse order, and for each operation do its inverse, and that takes B to A. Thus A is row equivalent to B.

    (c) Follow the operations which take A to B with those which take B to C.

12. (a) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & \frac{5}{3} & 1 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

Section 2.2, p. 113

2. (a) $x = -6 - s - t$, $y = s$, $z = t$, $w = 5$.
   (b) $x = -3$, $y = -2$, $z = 1$.
4. (a) $x = 5 + 2t$, $y = 2 - t$, $z = t$.
   (b) $x = 1$, $y = 2$, $z = 4 + t$, $w = t$.
6. (a) $x = -2 + r$, $y = -1$, $z = 8 - 2r$, $x_4 = r$, where r is any real number.
   (b) $x = 1$, $y = \frac{2}{3}$, $z = -\frac{2}{3}$.
   (c) No solution.
8. (a) $x = 1 - r$, $y = 2$, $z = 1$, $x_4 = r$, where r is any real number.
   (b) $x = 1 - r$, $y = 2 + r$, $z = -1 + r$, $x_4 = r$, where r is any real number.
10. $\mathbf{x} = \begin{bmatrix} r \\ 0 \end{bmatrix}$, where $r \neq 0$.
12. $\mathbf{x} = \begin{bmatrix} -\frac{1}{4}r \\ \frac{1}{4}r \\ r \end{bmatrix}$, where $r \neq 0$.
14. (a) $a = -2$. (b) $a \neq \pm 2$. (c) $a = 2$.
16. (a) $a = \pm\sqrt{6}$. (b) $a \neq \pm\sqrt{6}$.
18. The augmented matrix is $\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \end{array}\right]$. If we reduce this matrix to reduced row echelon form, we see that the linear system has only the trivial solution if and only if A is row equivalent to I_2. Now show that this occurs if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$ then at least one of a or c is $\neq 0$, and it is a routine matter to show that A is row equivalent to I_2. If $ad - bc = 0$, then by case considerations we find that A is row equivalent to a matrix that has a row or column consisting entirely of zeros, so that A is not row equivalent to I_2.

    Alternate proof: If $ad - bc \neq 0$, then A is nonsingular, so the only solution is the trivial one. If $ad - bc = 0$, then $ad = bc$. If $ad = 0$ then either a or $d = 0$, say $a = 0$. Then $bc = 0$, and either b or $c = 0$. In any of these cases we get a nontrivial solution. If $ad \neq 0$, then $\frac{a}{c} = \frac{b}{d}$, and the second equation is a multiple of the first one so we again have a nontrivial solution.
19. This had to be shown in the first proof of Exercise 18 above. If the alternate proof of Exercise 18 was given, then Exercise 19 follows from the former by noting that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if A is row equivalent to I_2 and this occurs if and only if $ad - bc \neq 0$.
20. $\begin{bmatrix} \frac{3}{2} \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} t$, where t is any number.
22. $-a + b + c = 0$.
24. (a) Change "row" to "column."
    (b) Proceed as in the proof of Theorem 2.1, changing "row" to "column."

25. Using Exercise 24(b) we can assume that every $m \times n$ matrix A is column equivalent to a matrix in column echelon form. That is, A is column equivalent to a matrix B that satisfies the following:

    (a) All columns consisting entirely of zeros, if any, are at the right side of the matrix.

    (b) The first nonzero entry in each column that is not all zeros is a 1, called the leading entry of the column.

    (c) If the columns j and $j+1$ are two successive columns that are not all zeros, then the leading entry of column $j+1$ is below the leading entry of column j.

    We start with matrix B and show that it is possible to find a matrix C that is column equivalent to B that satisfies

    (d) If a row contains a leading entry of some column then all other entries in that row are zero.

    If column j of B contains a nonzero element, then its first (counting top to bottom) nonzero element is a 1. Suppose the 1 appears in row r_j. We can perform column operations of the form $ac_j + c_k$ for each of the nonzero columns c_k of B such that the resulting matrix has row r_j with a 1 in the (r_j, j) entry and zeros everywhere else. This can be done for each column that contains a nonzero entry hence we can produce a matrix C satisfying (d). It follows that C is the unique matrix in reduced column echelon form and column equivalent to the original matrix A.

26. $-3a - b + c = 0$.

28. Apply Exercise 18 to the linear system given here. The coefficient matrix is

$$\begin{bmatrix} a-r & d \\ c & b-r \end{bmatrix}.$$

    Hence from Exercise 18, we have a nontrivial solution if and only if $(a-r)(b-r) - cd = 0$.

29. (a) $A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

    (b) Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x} = \mathbf{b}$ and let $\mathbf{x}$ be any solution to $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$. Then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = \mathbf{x}_p + (\mathbf{x} - \mathbf{x}_p)$ and $A\mathbf{x}_h = A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Thus $\mathbf{x}_h$ is in fact a solution to $A\mathbf{x} = \mathbf{0}$.

30. (a) $3x^2 + 2$ (b) $2x^2 - x - 1$

32. $\frac{3}{2}x^2 - x + \frac{1}{2}$.

34. (a) $x = 0$, $y = 0$ (b) $x = 5$, $y = -7$

36. $r = 5$, $r_2 = 5$.

37. The GPS receiver is located at the tangent point where the two circles intersect.

38. $4\mathrm{Fe} + 3\mathrm{O}_2 \rightarrow 2\mathrm{Fe}_2\mathrm{O}_3$

40. $\mathbf{x} = \begin{bmatrix} 0 \\ \frac{1}{4} - \frac{1}{4}i \end{bmatrix}$.

42. No solution.

Section 2.3, p. 124

1. The elementary matrix E which results from I_n by a type I interchange of the ith and jth row differs from I_n by having 1's in the (i,j) and (j,i) positions and 0's in the (i,i) and (j,j) positions. For that E, EA has as its ith row the jth row of A and for its jth row the ith row of A.

   The elementary matrix E which results from I_n by a type II operation differs from I_n by having $c \neq 0$ in the (i,i) position. Then EA has as its ith row c times the ith row of A.

   The elementary matrix E which results from I_n by a type III operation differs from I_n by having c in the (j,i) position. Then EA has as jth row the sum of the jth row of A and c times the ith row of A.

2. (a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

4. (a) Add 2 times row 1 to row 3: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C$

   (b) Add 2 times row 1 to row 3: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = B$

   (c) $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   Therefore B is the inverse of A.

6. If E_1 is an elementary matrix of type I then $E_1^{-1} = E_1$. Let E_2 be obtained from I_n by multiplying the ith row of I_n by $c \neq 0$. Let E_2^* be obtained from I_n by multiplying the ith row of I_n by $\frac{1}{c}$. Then $E_2 E_2^* = I_n$. Let E_3 be obtained from I_n by adding c times the ith row of I_n to the jth row of I_n. Let E_3^* be obtained from I_n by adding $-c$ times the ith row of I_n to the jth row of I_n. Then $E_3 E_3^* = I_n$.

8. $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ -1 & 0 & 1 \end{bmatrix}$.

10. (a) Singular. (b) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ -\frac{3}{2} & \frac{5}{2} & -\frac{1}{2} \end{bmatrix}$. (c) $\begin{bmatrix} -1 & \frac{3}{2} & \frac{1}{2} \\ 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. (d) $\begin{bmatrix} \frac{3}{5} & -\frac{3}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{3}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$.

12. (a) $A^{-1} = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{5} & 1 & \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{2} & -\frac{2}{5} & -\frac{1}{5} \end{bmatrix}$. (b) Singular.

14. A is row equivalent to I_3; a possible answer is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

16. $A = \begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix}.$

18. (b) and (c).

20. For $a = -1$ or $a = 3$.

21. This follows directly from Exercise 19 of Section 2.1 and Corollary 2.2. To show that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we proceed as follows:

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & db - bd \\ -ca + ac & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

22. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

23. The matrices A and B are row equivalent if and only if $B = E_k E_{k-1} \cdots E_2 E_1 A$. Let $P = E_k E_{k-1} \cdots E_2 E_1$.

24. If A and B are row equivalent then $B = PA$, where P is nonsingular, and $A = P^{-1}B$ (Exercise 23). If A is nonsingular then B is nonsingular, and conversely.

25. Suppose B is singular. Then by Theorem 2.9 there exists $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$. Then $(AB)\mathbf{x} = A\mathbf{0} = \mathbf{0}$, which means that the homogeneous system $(AB)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Theorem 2.9 implies that AB is singular, a contradiction. Hence, B is nonsingular. Since $A = (AB)B^{-1}$ is a product of nonsingular matrices, it follows that A is nonsingular.

    Alternate Proof: If AB is nonsingular it follows that AB is row equivalent to I_n, so $P(AB) = I_n$. Since P is nonsingular, $P = E_k E_{k-1} \cdots E_2 E_1$. Then $(PA)B = I_n$ or $(E_k E_{k-1} \cdots E_2 E_1 A)B = I_n$. Letting $E_k E_{k-1} \cdots E_2 E_1 A = C$, we have $CB = I_n$, which implies that B is nonsingular. Since $PAB = I_n$, $A = P^{-1}B^{-1}$, so A is nonsingular.

26. The matrix A is row equivalent to O if and only if $A = PO = O$ where P is nonsingular.

27. The matrix A is row equivalent to B if and only if $B = PA$, where P is a nonsingular matrix. Now $B^T = A^T P^T$, so A is row equivalent to B if and only if A^T is column equivalent to B^T.

28. If A has a row of zeros, then A cannot be row equivalent to I_n, and so by Corollary 2.2, A is singular. If the jth column of A is the zero column, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, the vector $\mathbf{x}$ with 1 in the jth entry and zeros elsewhere. By Theorem 2.9, A is singular.

29. (a) No. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $(A + B)^{-1}$ exists but A^{-1} and B^{-1} do not. Even supposing they all exist, equality need not hold. Let $A = \begin{bmatrix} 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \end{bmatrix}$ so $(A + B)^{-1} = \begin{bmatrix} \frac{1}{3} \end{bmatrix} \neq \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \end{bmatrix} = A^{-1} + B^{-1}$.

(b) Yes, for A nonsingular and $r \neq 0$.

$$(rA)\left[\frac{1}{r}A^{-1}\right] = r\left[\frac{1}{r}\right]A \cdot A^{-1} = 1 \cdot I_n = I_n.$$

30. Suppose that A is nonsingular. Then $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = A^{-1}\mathbf{b}$ for every $n \times 1$ matrix $\mathbf{b}$. Conversely, suppose that $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$. Letting $\mathbf{b}$ be the matrices

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

we see that we have solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ to the linear systems

$$A\mathbf{x}_1 = \mathbf{e}_1, \quad A\mathbf{x}_2 = \mathbf{e}_2, \quad \ldots, \quad A\mathbf{x}_n = \mathbf{e}_n. \tag{$*$}$$

Letting C be the matrix whose jth column is $\mathbf{x}_j$, we can write the n systems in $(*)$ as $AC = I_n$, since $I_n = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$. Hence, A is nonsingular.

31. We consider the case that A is nonsingular and upper triangular. A similar argument can be given for A lower triangular.

By Theorem 2.8, A is a product of elementary matrices which are the inverses of the elementary matrices that "reduce" A to I_n. That is,

$$A = E_1^{-1} \cdots E_k^{-1}.$$

The elementary matrix E_i will be upper triangular since it is used to introduce zeros into the upper triangular part of A in the reduction process. The inverse of E_i is an elementary matrix of the same type and also an upper triangular matrix. Since the product of upper triangular matrices is upper triangular and we have $A^{-1} = E_k \cdots E_1$ we conclude that A^{-1} is upper triangular.

Section 2.4, p. 129

1. See the answer to Exercise 4, Section 2.1. Where it mentions only row operations, now read "row and column operations".

2. (a) $\begin{bmatrix} I_4 \\ 0 \end{bmatrix}$. (b) I_3. (c) $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$. (d) I_4.

4. Allowable equivalence operations ("elementary row or elementary column operation") include in particular elementary row operations.

5. A and B are equivalent if and only if $B = E_t \cdots E_2 E_1 A F_1 F_2 \cdots F_s$. Let $E_t E_{t-1} \cdots E_2 E_1 = P$ and $F_1 F_2 \cdots F_s = Q$.

6. $B = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$; a possible answer is: $B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

8. Suppose A were nonzero but equivalent to O. Then some ultimate elementary row or column operation must have transformed a nonzero matrix A_r into the zero matrix O. By considering the types of elementary operations we see that this is impossible.

9. Replace "row" by "column" and vice versa in the elementary operations which transform A into B.

10. Possible answers are:
(a) $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -1 & 4 & 3 \\ 0 & 2 & -5 & -2 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 5 & 5 & 4 & 4 \end{bmatrix}$.

11. If A and B are equivalent then $B = PAQ$ and $A = P^{-1}BQ^{-1}$. If A is nonsingular then B is nonsingular, and conversely.

Section 2.5, p. 136

2. $\mathbf{x} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$.

4. $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 5 \end{bmatrix}$.

6. $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} -3 & 1 & -2 \\ 0 & 6 & 2 \\ 0 & 0 & -4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$.

8. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ -2 & 3 & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} -5 & 4 & 0 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 5 \\ -4 \end{bmatrix}$.

10. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 1 & 0 & 0 \\ -0.4 & 0.8 & 1 & 0 \\ 2 & -1.2 & -0.4 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 4 & 1 & 0.25 & -0.5 \\ 0 & 0.4 & 1.2 & -2.5 \\ 0 & 0 & -0.85 & 2 \\ 0 & 0 & 0 & -2.5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1.5 \\ 4.2 \\ 2.6 \\ -2 \end{bmatrix}$.

Supplementary Exercises for Chapter 2, p. 137

2. (a) $a = -4$ or $a = 2$.
   (b) The system has a solution for each value of a.

4. $c + 2a - 3b = 0$.

5. (a) Multiply the jth row of B by $\frac{1}{k}$.
   (b) Interchange the ith and jth rows of B.
   (c) Add $-k$ times the jth row of B to its ith row.

6. (a) If we transform E_1 to reduced row echelon form, we obtain I_n. Hence E_1 is row equivalent to I_n and thus is nonsingular.
   (b) If we transform E_2 to reduced row echelon form, we obtain I_n. Hence E_2 is row equivalent to I_n and thus is nonsingular.

(c) If we transform E_3 to reduced row echelon form, we obtain I_n. Hence E_3 is row equivalent to I_n and thus is nonsingular.

8. $\begin{bmatrix} 1 & -a & a^2 & -a^3 \\ 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

10. (a) $\begin{bmatrix} -41 \\ 47 \\ -35 \end{bmatrix}$. (b) $\begin{bmatrix} 83 \\ -45 \\ -62 \end{bmatrix}$.

12. $s \neq 0, \pm\sqrt{2}$.

13. For any angle θ, $\cos\theta$ and $\sin\theta$ are never simultaneously zero. Thus at least one element in column 1 is not zero. Assume $\cos\theta \neq 0$. (If $\cos\theta = 0$, then interchange rows 1 and 2 and proceed in a similar manner to that described below.) To show that the matrix is nonsingular and determine its inverse, we put

$$\left[\begin{array}{cc|cc} \cos\theta & \sin\theta & 1 & 0 \\ -\sin\theta & \cos\theta & 0 & 1 \end{array}\right]$$

into reduced row echelon form. Apply row operations $\frac{1}{\cos\theta}$ times row 1 and $\sin\theta$ times row 1 added to row 2 to obtain

$$\left[\begin{array}{cc|cc} 1 & \dfrac{\sin\theta}{\cos\theta} & \dfrac{1}{\cos\theta} & 0 \\ & & & \\ 0 & \dfrac{\sin^2\theta}{\cos\theta} + \cos\theta & \dfrac{\sin\theta}{\cos\theta} & 1 \end{array}\right].$$

Since

$$\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta},$$

the $(2,2)$-element is not zero. Applying row operations $\cos\theta$ times row 2 and $\left(-\frac{\sin\theta}{\cos\theta}\right)$ times row 2 added to row 1 we obtain

$$\left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & -\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right].$$

It follows that the matrix is nonsingular and its inverse is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

14. (a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
    (b) $A(\mathbf{u} - \mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{0} - \mathbf{0} = \mathbf{0}$.
    (c) $A(r\mathbf{u}) = r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0}$.
    (d) $A(r\mathbf{u} + s\mathbf{v}) = r(A\mathbf{u}) + s(A\mathbf{v}) = r\mathbf{0} + s\mathbf{0} = \mathbf{0}$.

15. If $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$, then $A(\mathbf{u} - \mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

16. Suppose at some point in the process of reducing the augmented matrix to reduced row echelon form we encounter a row whose first n entries are zero but whose $(n+1)$st entry is some number $c \neq 0$. The corresponding linear equation is

$$0 \cdot x_1 + \cdots + 0 \cdot x_n = c \quad \text{or} \quad 0 = c.$$

This equation has no solution, thus the linear system is inconsistent.

17. Let $\mathbf{u}$ be one solution to $A\mathbf{x} = \mathbf{b}$. Since A is singular, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{u}_0$. Then for any real number r, $\mathbf{v} = r\mathbf{u}_0$ is also a solution to the homogeneous system. Finally, by Exercise 29, Sec. 2.2, for each of the infinitely many vectors $\mathbf{v}$, the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is a solution to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$.

18. $s = 1$, $t = 1$.

20. If any of the diagonal entries of L or U is zero, there will not be a unique solution.

21. The outer product of X and Y can be written in the form

$$XY^T = \begin{bmatrix} x_1 \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \\ x_2 \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \\ \vdots \\ x_n \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \end{bmatrix}.$$

If either $X = O$ or $Y = O$, then $XY^T = O$. Thus assume that there is at least one nonzero component in X, say x_i, and at least one nonzero component in Y, say y_j. Then $\left(\frac{1}{x_i}\right)\text{Row}_i(XY^T)$ makes the ith row exactly Y^T. Since all the other rows are multiples of Y^T, row operations of the form $-x_kR_i + R_p$, for $p \neq i$, can be performed to zero out everything but the ith row. It follows that either XY^T is row equivalent to O or to a matrix with $n-1$ zero rows.

Chapter Review for Chapter 2, p. 138

True or False

1. False.
2. True.
3. False.
4. True.
5. True.
6. True.
7. True.
8. True.
9. True.
10. False.

Quiz

1. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

2. (a) No.
   (b) Infinitely many.
   (c) No.
   (d) $\mathbf{x} = \begin{bmatrix} -6 + 2r + 7s \\ r \\ -3s \\ s \end{bmatrix}$, where r and s are any real numbers.

3. $k = 6$.

4. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

5. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

6. $P = A^{-1}$, $Q = B$.

7. Possible answers: Diagonal, zero, or symmetric.

Chapter 3

Determinants

Section 3.1, p. 145

2. (a) 4. (b) 7. (c) 0.

4. (a) odd. (b) even. (c) even.

6. (a) $-$. (b) $+$. (c) $+$.

8. (a) 7. (b) 2.

10. $\det(A) = a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} - a_{11}a_{24}a_{33}a_{42} + \cdots$ (24 summands).

12. (a) -24. (b) -36. (c) 180.

14. (a) $t^2 - 8t - 20$. (b) $t^3 - t$.

16. (a) $t = 10$, $t = -2$. (b) $t = 0$, $t = 1$, $t = -1$.

Section 3.2, p. 154

2. (a) 4. (b) -24. (c) -30. (d) 72. (e) -120. (f) 0.

4. -2.

6. (a) $\det(A) = -7$, $\det(B) = 3$. (b) $\det(A) = -24$, $\det(B) = -30$.

8. Yes, since $\det(AB) = \det(A)\det(B)$ and $\det(BA) = \det(B)\det(A)$.

9. Yes, since $\det(AB) = \det(A)\det(B)$ implies that $\det(A) = 0$ or $\det(B) = 0$.

10. $\det(cA) = \sum(\pm)(ca_{1j_1})(ca_{2j_2})\cdots(ca_{nj_n}) = c^n \sum(\pm)a_{1j_1}a_{2j_2}\cdots a_{nj_n} = c^n \det(A)$.

11. Since A is skew symmetric, $A^T = -A$. Therefore

$$\begin{aligned} \det(A) &= \det(A^T) && \text{by Theorem 3.1} \\ &= \det(-A) && \text{since } A \text{ is skew symmetric} \\ &= (-1)^n \det(A) && \text{by Exercise 10} \\ &= -\det(A) && \text{since } n \text{ is odd} \end{aligned}$$

The only number equal to its negative is zero, so $\det(A) = 0$.

12. This result follows from the observation that each term in $\det(A)$ is a product of n entries of A, each with its appropriate sign, with exactly one entry from each row and exactly one entry from each column.

13. We have $\det(AB^{-1}) = (\det A)(\det B^{-1}) = (\det A)\left(\dfrac{1}{\det B}\right)$.

14. If $AB = I_n$, then $\det(AB) = \det(A)\det(B) = \det(I_n) = 1$, so $\det(A) \neq 0$ and $\det(B) \neq 0$.

15. (a) By Corollary 3.3, $\det(A^{-1}) = 1/\det(A)$. Since $A = A^{-1}$, we have

$$\det(A) = \frac{1}{\det(A)} \quad \Longrightarrow \quad (\det(A))^2 = 1.$$

Hence $\det(A) = \pm 1$.

(b) If $A^T = A^{-1}$, then $\det(A^T) = \det(A^{-1})$. But

$$\det(A) = \det(A^T) \quad \text{and} \quad \det(A^{-1}) = \frac{1}{\det(A)}$$

hence we have

$$\det(A) = \frac{1}{\det(A)} \quad \Longrightarrow \quad (\det(A))^2 = 1 \quad \Longrightarrow \quad \det(A) = \pm 1.$$

16. From Definition 3.2, the only time we get terms which do not contain a zero factor is when the terms involved come from A and B alone. Each one of the column permutations of terms from A can be associated with every one of the column permutations of B. Hence by factoring we have

$$\begin{aligned}
\det\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\right) &= \sum(\text{terms from } A \text{ for any column permutation})|B| \\
&= |B|\sum(\text{terms from } A \text{ for any column permutation}) \\
&= (\det B)(\det A) = (\det A)(\det B).
\end{aligned}$$

17. If $A^2 = A$, then $\det(A^2) = [\det(A)]^2 = \det(A)$, so $\det(A) = 1$. Alternate solution: If $A^2 = A$ and A is nonsingular, then $A^{-1}A^2 = A^{-1}A = I_n$, so $A = I_n$ and $\det(A) = \det(I_n) = 1$.

18. Since $AA^{-1} = I_n$, $\det(AA^{-1}) = \det(I_n) = 1$, so $\det(A)\det(A^{-1}) = 1$. Hence, $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

19. From Definition 3.2, the only time we get terms which do not contain a zero factor is when the terms involved come from A and B alone. Each one of the column permutations of terms from A can be associated with every one of the column permutations of B. Hence by factoring we have

$$\begin{aligned}
\begin{vmatrix} A & O \\ C & B \end{vmatrix} &= \sum(\text{terms from } A \text{ for any column permutations})|B| \\
&= |B|\sum(\text{terms from } A \text{ for any column permutation}) \\
&= |B||A|
\end{aligned}$$

20. (a) $\det(A^TB^T) = \det(A^T)\det(B^T) = \det(A)\det(B^T)$.

(b) $\det(A^TB^T) = \det(A^T)\det(B^T) = \det(A^T)\det(B)$.

22. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$
$= (b-a)(c^2-a^2) - (c-a)(b^2-a^2) = (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a)$
$= (b-a)(c-a)[(c+a)-(b+a)] = (b-a)(c-a)(c-b)$.

24. (a) and (b).
26. (a) $t \neq 0$. (b) $t \neq \pm 1$. (c) $t \neq 0, \pm 1$.
28. The system has only the trivial solution.
29. If $A = \left[a_{ij}\right]$ is upper triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$, so $\det(A) \neq 0$ if and only if $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$.
30. (a) I_3
    (b) Only the trivial solution.
31. (a) A matrix having at least one row of zeros.
    (b) Infinitely many.
32. If $A^2 = A$, then $\det(A^2) = \det(A)$, so $[\det(A)]^2 = \det(A)$. Thus, $\det(A)(\det(A) - 1) = 0$. This implies that $\det(A) = 0$ or $\det(A) = 1$.
33. If A and B are similar, then there exists a nonsingular matrix P such that $B = P^{-1}AP$. Then
$$\det(B) = \det(P^{-1}BP) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)}\det(P)\det(A) = \det(A).$$
34. If $\det(A) \neq 0$, then A is nonsingular. Hence, $A^{-1}AB = A^{-1}AC$, so $B = C$.
36. In MATLAB the command for the determinant actually invokes an LU-factorization, hence is closely associated with the material in Section 2.5.
37. For $\epsilon = 10^{-5}$, MATLAB gives the determinant as -3×10^{-5} which agrees with the theory; for $\epsilon = 10^{-14}$, -3.2026×10^{-14}; for $\epsilon = 10^{-15}$, -6.2800×10^{-15}; for $\epsilon = 10^{-16}$, zero.

Section 3.3, p. 164

2. (a) -23. (b) 7. (c) 15. (d) -28.
4. (a) -3. (b) 0. (c) 3. (d) 6.
6. (b) 2. (c) 24. (f) -30.
8. (b) -24. (d) 72. (e) -120.
9. We proceed by successive expansions along first columns:
$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22}\cdots a_{nn}.$$
12. $t = 1$, $t = -1$, $t = -2$.
13. (a) From Definition 3.2 each term in the expansion of the determinant of an $n \times n$ matrix is a product of n entries of the matrix. Each of these products contains exactly one entry from each row and exactly one entry from each column. Thus each such product from $\det(tI_n - A)$ contains at most n terms of the form $t - a_{ii}$. Hence each of these products is at most a polynomial of degree n. Since one of the products has the form $(t - a_{11})(t - a_{22})\cdots(t - a_{nn})$ it follows that the sum of the products is a polynomial of degree n in t.

(b) The coefficient of t^n is 1 since it only appears in the term $(t-a_{11})(t-a_{22})\cdots(t-a_{nn})$ which we discussed in part (a). (The permutation of the column indices is even here so a plus sign is associated with this term.)

(c) Using part (a), suppose that

$$\det(tI_n - A) = t^n + c_1t^{n-1} + c_2t^{n-2} + \cdots + c_{n-1}t + c_n.$$

Set $t=0$ and we have $\det(-A) = c_n$ which implies that $c_n = (-1)^n\det(A)$. (See Exercise 10 in Section 6.2.)

14. (a) $f(t) = t^2 - 5t - 2$, $\det(A) = -2$.

(b) $f(t) = t^3 - t^2 - 13t - 26$, $\det(A) = 26$.

(c) $f(t) = t^2 - 2t$, $\det(A) = 0$.

16. 6.

18. Let $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_3(x_3,y_3)$ be the vertices of a triangle T. Then from Equation (2), we have

$$\text{area of } T = \frac{1}{2}\left|\det\left(\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right)\right| = \tfrac{1}{2}\left|x_1y_2 + y_1x_3 + x_2y_3 - x_3y_2 - y_3x_1 - x_2y_1\right|.$$

Let A be the matrix representing a counterclockwise rotation L through an angle ϕ. Thus

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

and P_1', P_2', P_3' are the vertices of $L(T)$, the image of T. We have

$$L\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = \begin{bmatrix} x_1\cos\phi - y_1\sin\phi \\ x_1\sin\phi + y_1\cos\phi \end{bmatrix},$$
$$L\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_2\cos\phi - y_2\sin\phi \\ x_2\sin\phi + y_2\cos\phi \end{bmatrix},$$
$$L\left(\begin{bmatrix} x_3 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} x_3\cos\phi - y_3\sin\phi \\ x_3\sin\phi + y_3\cos\phi \end{bmatrix},$$

Then

$$\begin{aligned}\text{area of } L(T) &= \frac{1}{2}\left|\det\left(\begin{bmatrix} x_1\cos\phi - y_1\sin\phi & x_1\sin\phi + y_1\cos\phi & 1 \\ x_2\cos\phi - y_2\sin\phi & x_2\sin\phi + y_2\cos\phi & 1 \\ x_3\cos\phi - y_3\sin\phi & x_3\sin\phi + y_3\cos\phi & 1 \end{bmatrix}\right)\right| \\ &= \tfrac{1}{2}\left|(x_1\cos\phi - y_1\sin\phi)[x_2\sin\phi + y_2\cos\phi - x_3\sin\phi - y_3\cos\phi]\right. \\ &\qquad + (x_2\cos\phi - y_2\sin\phi)[x_3\sin\phi + y_3\cos\phi - x_1\sin\phi - y_1\cos\phi] \\ &\qquad \left. + (x_3\cos\phi - y_3\sin\phi)[x_1\sin\phi + y_1\cos\phi - x_2\sin\phi - y_2\cos\phi]\right| \\ &= \tfrac{1}{2}\left|x_1y_2 + y_1x_3 + x_2y_3 - x_3y_2 - x_1y_3 - x_2y_1\right| \\ &= \text{area of } T.\end{aligned}$$

19. Let T be the triangle with vertices (x_1,y_1), (x_2,y_2), and (x_3,y_3). Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and define the linear operator $L\colon R^2 \to R^2$ by $L(\mathbf{v}) = A\mathbf{v}$ for $\mathbf{v}$ in R^2. The vertices of $L(T)$ are

$$(ax_1 + by_1, cx_1 + dy_1), \quad (ax_2 + by_2, cx_2 + dy_2), \quad \text{and} \quad (ax_3 + by_3, cx_3 + dy_3).$$

Then by Equation (2),

$$\text{Area of } T = \tfrac{1}{2}\,|x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2|$$

and

$$\begin{aligned}\text{Area of } L(T) = \tfrac{1}{2}\,|&ax_1dy_2 - ax_1dy_3 - ax_2dy_1 + ax_2dy_3 + ax_3dy_1 - ax_3dy_2\\ &-bcx_1y_2 + bcx_1y_3 + bcx_2y_1 - bcx_2y_3 - bcx_3y_1 + bcx_3y_2|\end{aligned}$$

Now,

$$\begin{aligned}|\det(A)| \cdot \text{Area of } T &= |ad - bc|\,\tfrac{1}{2}\,|x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2|\\ &= \tfrac{1}{2}\,|ax_1dy_2 - ax_1dy_3 - ax_2dy_1 + ax_2dy_3 + ax_3dy_1 - ax_3dy_2\\ &\qquad -bcx_1y_2 + bcx_1y_3 + bcx_2y_1 - bcx_2y_3 - bcx_3y_1 + bcx_3y_2|\\ &= |\text{Area of } L(T)|\end{aligned}$$

Section 3.4, p. 169

2. (a) $\begin{bmatrix} 2 & -7 & -6 \\ 1 & -7 & -3 \\ -4 & 7 & 5 \end{bmatrix}$. (b) -7.

4. $\begin{bmatrix} -\frac{2}{7} & 1 & \frac{6}{7} \\ -\frac{1}{7} & 1 & \frac{3}{7} \\ \frac{4}{7} & -1 & -\frac{5}{7} \end{bmatrix}$.

6. If A is symmetric, then for each i and j, M_{ji} is the transpose of M_{ij}. Thus $A_{ji} = (-1)^{j+i}|M_{ji}| = (-1)^{i+j}|M_{ij}| = A_{ij}$.

8. The adjoint matrix is upper triangular if A is upper triangular, since $a_{ij} = 0$ if $i > j$ which implies that $A_{ij} = 0$ if $i > j$.

10. $\dfrac{1}{(b-a)(c-a)(c-b)} \begin{bmatrix} bc(c-b) & ac(a-c) & ab(b-a) \\ b^2 - c^2 & c^2 - a^2 & a^2 - b^2 \\ c - b & a - c & b - a \end{bmatrix}$.

12. $-\dfrac{1}{24} \begin{bmatrix} -6 & -2 & 9 \\ 0 & 8 & -12 \\ 0 & 0 & -12 \end{bmatrix}$.

13. We follow the hint. If A is singular then $\det(A) = 0$. Hence $A(\text{adj } A) = \det(A)\, I_n = 0I_n = O$. If adj A were nonsingular, $(\text{adj } A)^{-1}$ exists. Then we have

$$A(\text{adj } A)(\text{adj } A)^{-1} = A = O(\text{adj } A)^{-1} = O,$$

that is, $A = O$. But the adjoint of the zero matrix must be a matrix of all zeros. Thus adj $A = O$ so adj A is singular. This is a contradiction. Hence it follows that adj A is singular.

14. If A is singular, then adj A is also singular by Exercise 13, and $\det(\text{adj}\, A) = 0 = [\det(A)]^{n-1}$. If A is nonsingular, then $A(\text{adj}\, A) = \det(A)I_n$. Taking the determinant on each side,

$$\det(A)\det(\text{adj}\, A) = \det(\det(A)I_n) = [\det(A)]^n.$$

Thus $\det(\text{adj}\, A) = [\det(A)]^{n-1}$.

Section 3.5, p. 172

2. $x_1 = 1$, $x_2 = -1$, $x_3 = 0$, $x_4 = 2$.
4. $x_1 = 1$, $x_2 = 2$, $x_3 = -2$.
6. $x_1 = 1$, $x_2 = \frac{2}{3}$, $x_3 = -\frac{2}{3}$.

Supplementary Exercises for Chapter 3, p. 174

2. (a) $t = 1, 4$. (b) $t = 3, 4, -1$. (c) $t = 1, 2, 3$. (d) $t = -3, 1, -1$.
3. If $A^n = O$ for some positive integer n, then

$$0 = \det(O) = \det(A^n) = \det\underbrace{(A\,A\cdots A)}_{n \text{ times}} = \underbrace{\det(A)\det(A)\cdots\det(A)}_{n \text{ times}} = (\det(A))^n.$$

It follows that $\det(A) = 0$.

4. (a) $\begin{vmatrix} a & 1 & b \\ b & 1 & c \\ c & 1 & a \end{vmatrix}_{-1\mathbf{c}_3+\mathbf{c}_1\to\mathbf{c}_1} = \begin{vmatrix} a-b & 1 & b \\ b-c & 1 & c \\ c-a & 1 & a \end{vmatrix}_{\mathbf{c}_1+\mathbf{c}_3\to\mathbf{c}_3} = \begin{vmatrix} a-b & 1 & a \\ b-c & 1 & b \\ c-a & 1 & c \end{vmatrix}$

(b) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}_{-\mathbf{r}_1+\mathbf{r}_2\to\mathbf{r}_2;\ -\mathbf{r}_1+\mathbf{r}_3\to\mathbf{r}_3} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b+a)(b-a) \\ 0 & c-a & (c+a)(c-a) \end{vmatrix}_{-a\mathbf{c}_1+\mathbf{c}_2\to\mathbf{c}_2;\ -a^2\mathbf{c}_1+\mathbf{c}_3\to\mathbf{c}_3}$

$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & b-a & (b+a)(b-a) \\ 0 & c-a & (c+a)(c-a) \end{vmatrix}_{-(a+b+c)\mathbf{c}_2+\mathbf{c}_3\to\mathbf{c}_3} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b-a & -c(b-a) \\ 0 & c-a & -b(c-a) \end{vmatrix}_{a\mathbf{c}_1+\mathbf{c}_2\to\mathbf{c}_2;\ bc\mathbf{c}_1+\mathbf{c}_3\to\mathbf{c}_3}$

$= \begin{vmatrix} 1 & a & bc \\ 0 & b-a & -c(b-a) \\ 0 & c-a & -b(c-a) \end{vmatrix}_{\mathbf{r}_1+\mathbf{r}_2\to\mathbf{r}_2;\ \mathbf{r}_1+\mathbf{r}_3\to\mathbf{r}_3} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}.$

5. If A is an $n \times n$ matrix then

$$\det(AA^T) = \det(A)\ \det(A^T) = \det(A)\ \det(A) = (\det(A))^2.$$

(Here we used Theorems 3.9 and 3.1.) Since the square of any real number is ≥ 0 we have $\det(AA^T) \geq 0$.

6. The determinant is not a linear transformation from R_{nn} to R^1 for $n > 1$ since for an arbitrary scalar c, $\det(cA) = c^n \det(A) \neq c\det(A)$.
7. Since A is nonsingular, Corollary 3.4 implies that

$$A^{-1} = \frac{1}{\det(A)}\,(\text{adj } A).$$

Multiplying both sides on the left by A gives

$$AA^{-1} = I_n = \frac{1}{\det(A)}\,A\,(\text{adj } A).$$

Hence we have that

$$(\text{adj } A)^{-1} = \frac{1}{\det(A)}\,A.$$

From Corollary 3.4 it follows that for any nonsingular matrix B, adj $B = \det(B)\,B^{-1}$. Let $B = A^{-1}$ and we have

$$\text{adj }(A^{-1}) = \det(A^{-1})\,(A^{-1})^{-1} = \frac{1}{\det(A)}\,A = (\text{adj } A)^{-1}.$$

8. If rows i and j are proportional with $ta_{ik} = a_{jk}$, $k = 1, 2, \dots, n$, then

$$\det(A) = \det(A)_{-t\mathbf{r}_i + \mathbf{r}_j \to \mathbf{r}_j} = 0$$

since this row operation makes row j all zeros.

9. Matrix Q is $n \times n$ with each entry equal to 1. Then, adding row j to row 1 for $j = 2, 3, \dots, n$, we have

$$\det(Q - nI_n) = \begin{vmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1-n & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-n \end{vmatrix} = 0$$

by Theorem 3.4.

10. If A has integer entries then the cofactors of A are integers and adj A has only integer entries. If A is nonsingular and

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A$$

has integer entries it must follow that $\dfrac{1}{\det(A)}$ times each entry of adj A is an integer. Since adj A has integer entries $\dfrac{1}{\det(A)}$ must be an integer, so $\det(A) = \pm 1$. Conversely, if $\det(A) = \pm 1$, then A is nonsingular and $A^{-1} = \pm 1 \operatorname{adj} A$ implies that A^{-1} has integer entries.

11. If A and $\mathbf{b}$ have integer entries and $\det(A) = \pm 1$, then using Cramer's rule to solve $A\mathbf{x} = \mathbf{b}$, we find that the numerator in the fraction giving x_i is an integer and the denominator is ± 1, so x_i is an integer for $i = 1, 2, \dots, n$.

Chapter Review for Chapter 3, p. 174

True or False

1. False. 2. True. 3. False. 4. True. 5. True. 6. False.
7. False. 8. True. 9. True. 10. False. 11. True. 12. False.

Quiz

1. -54.

2. False.

3. -1.

4. -2.

5. Let the diagonal entries of A be $d_{11}, \dots, d_{nn}$. Then $\det(A) = d_{11} \cdots d_{nn}$. Since A is singular if and only if $\det(A) = 0$, A is singular if and only if some diagonal entry d_{ii} is zero.

6. 19.

7. $A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{5}{2} & 1 \\ 1 & -3 & -1 \\ -1 & 4 & 1 \end{bmatrix}$.

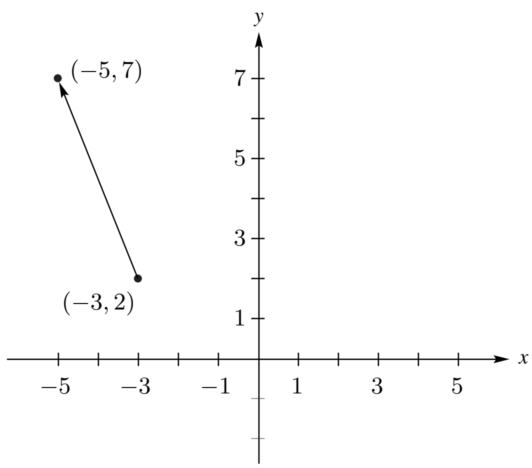
8. $\det(A) = 14$. Therefore $x_1 = \frac{11}{7}$, $x_2 = -\frac{4}{7}$, $x_3 = -\frac{5}{7}$.

Chapter 4

Real Vector Spaces

Section 4.1, p. 187

2. $(-5, 7)$.



4. $(1, -6, 3)$.

6. $a = -2$, $b = -2$, $c = -5$.

8. (a) $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$. (b) $\begin{bmatrix} 0 \\ -3 \\ -6 \end{bmatrix}$.

10. (a) $\begin{bmatrix} -4 \\ 7 \end{bmatrix}$. (b) $\begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$.

12. (a) $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$, $2\mathbf{u}-\mathbf{v} = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$, $3\mathbf{u}-2\mathbf{v} = \begin{bmatrix} -1 \\ 6 \\ 7 \end{bmatrix}$, $\mathbf{0}-3\mathbf{v} = \begin{bmatrix} -6 \\ 0 \\ -3 \end{bmatrix}$.

    (b) $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $2\mathbf{u}-\mathbf{v} = \begin{bmatrix} 3 \\ -4 \\ 11 \end{bmatrix}$, $3\mathbf{u}-2\mathbf{v} = \begin{bmatrix} 4 \\ -7 \\ 18 \end{bmatrix}$, $\mathbf{0}-3\mathbf{v} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$.

(c) $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $2\mathbf{u}-\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix}$, $3\mathbf{u}-2\mathbf{v} = \begin{bmatrix} 5 \\ -2 \\ -11 \end{bmatrix}$, $\mathbf{0}-3\mathbf{v} = \begin{bmatrix} 3 \\ -3 \\ -12 \end{bmatrix}$.

14. (a) $r = 2$. (b) $s = \frac{8}{3}$. (c) $r = 3$, $s = -2$.

16. $c_1 = 1$, $c_2 = -2$.

18. Impossible.

20. $c_1 = r$, $c_2 = s$, $c_3 = t$.

22. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, then $(-1)\mathbf{u} = \begin{bmatrix} -u_1 \\ -u_2 \\ -u_3 \end{bmatrix} = -\mathbf{u}$.

23. Parts 2–8 of Theorem 4.1 require that we show equality of certain vectors. Since the vectors are column matrices, this is equivalent to showing that corresponding entries of the matrices involved are equal. Hence instead of displaying the matrices we need only work with the matrix entries. Suppose $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are in R^3 with c and d real scalars. It follows that all the components of matrices involved will be real numbers, hence when appropriate we will use properties of real numbers.

(2)
$$(\mathbf{u}+(\mathbf{v}+\mathbf{w}))_i = u_i + (v_i + w_i)$$
$$((\mathbf{u}+\mathbf{v})+\mathbf{w})_i = (u_i + v_i) + w_i$$

Since real numbers $u_i + (v_i + w_i)$ and $(u_i + v_i) + w_i$ are equal for $i = 1, 2, 3$ we have $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$.

(3)
$$(\mathbf{u}+\mathbf{0})_i = u_i + 0$$
$$(\mathbf{0}+\mathbf{u})_i = 0 + u_i$$
$$(\mathbf{u})_i = u_i$$

Since real numbers $u_i + 0$, $0 + u_i$, and u_i are equal for $i = 1, 2, 3$ we have $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$.

(4)
$$(\mathbf{u}+(-\mathbf{u}))_i = u_i + (-u_i)$$
$$(\mathbf{0})_i = 0$$

Since real numbers $u_i + (-u_i)$ and 0 are equal for $i = 1, 2, 3$ we have $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$.

(5)
$$(c(\mathbf{u}+\mathbf{v}))_i = c(u_i + v_i)$$
$$(c\mathbf{u}+c\mathbf{v})_i = cu_i + cv_i$$

Since real numbers $c(u_i + v_i)$ and $cu_i + cv_i$ are equal for $i = 1, 2, 3$ we have $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$.

(6)
$$((c+d)\mathbf{u})_i = (c+d)u_i$$
$$(c\mathbf{u}+d\mathbf{u})_i = cu_i + du_i$$

Since real numbers $(c+d)u_i$ and $cu_i + du_i$ are equal for $i = 1, 2, 3$ we have $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$.

(7)
$$(c(d\mathbf{u}))_i = c(du_i)$$
$$((cd)\mathbf{u})_i = (cd)u_i$$

Since real numbers $c(du_i)$ and $(cd)u_i$ are equal for $i = 1, 2, 3$ we have $c(d\mathbf{u}) = (cd)\mathbf{u}$.

(8)
$$(1\mathbf{u})_i = 1u_i$$
$$(\mathbf{u})_i = u_i$$

Since real numbers $1u_i$ and u_i are equal for $i = 1, 2, 3$ we have $1\mathbf{u} = \mathbf{u}$.

The proof for vectors in R^2 is obtained by letting i be only 1 and 2.

Section 4.2, p. 196

1. (a) The polynomials $t^2 + t$ and $-t^2 - 1$ are in P_2, but their sum $(t^2 + t) + (-t^2 - 1) = t - 1$ is not in P_2.

   (b) No, since $0(t^2 + 1) = 0$ is not in P_2.

2. (a) No.

   (b) Yes.

   (c) $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

   (d) Yes. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$, then $abcd = 0$. Let $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. Then $A \oplus -A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $-A \in V$ since $(-a)(-b)(-c)(-d) = 0$.

   (e) No. V is not closed under scalar multiplication.

4. No, since V is not closed under scalar multiplication. For example, $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \in V$, but $\frac{1}{2} \odot \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin V$.

5. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$.

   (1) For each $i = 1, \dots, n$, the ith component of $\mathbf{u} + \mathbf{v}$ is $u_i + v_i$, which equals the ith component $v_i + u_i$ of $\mathbf{v} + \mathbf{u}$.

   (2) For each $i = 1, \dots, n$, $u_i + (v_i + w_i) = (u_i + v_i) + w_i$.

   (3) For each $i = 1, \dots, n$, $u_i + 0 = 0 + u_i = u_i$.

   (4) For each $i = 1, \dots, n$, $u_i + (-u_i) = (-u_i) + u_i = 0$.

   (5) For each $i = 1, \dots, n$, $c(u_i + v_i) = cu_i + cv_i$.

   (6) For each $i = 1, \dots, n$, $(c + d)u_i = cu_i + du_i$.

   (7) For each $i = 1, \dots, n$, $c(du_i) = (cd)u_i$.

   (8) For each $i = 1, \dots, n$, $1 \cdot u_i = u_i$.

6. P is a vector space.

   (a) Let $p(t)$ and $q(t)$ be polynomials not both zero. Suppose the larger of their degrees is n. Then $p(t) + q(t)$ and $cp(t)$ are computed as in Example 5. The properties of Definition 4.4 are verified as in Example 5.

8. Property 6.

10. Properties 4 and (b).

12. The vector $\mathbf{0}$ is the real number 1, and if $\mathbf{u}$ is a vector (that is, a positive real number) then $\mathbf{u}^{-1}$ is $\frac{1}{\mathbf{u}}$.

13. The vector $\mathbf{0}$ in V is the constant zero function.

14. Verify the properties in Definition 4.4.

15. Verify the properties in Definition 4.4.

16. No.

17. No. The zero element for $\oplus$ would have to be the real number 1, but then $\mathbf{u} = 0$ has no "negative" $\mathbf{v}$ such that $\mathbf{u} \oplus \mathbf{v} = 0 \cdot \mathbf{v} = 1$. Thus (4) fails to hold. (5) fails since $c \odot (\mathbf{u} \oplus \mathbf{v}) = c + (\mathbf{uv}) \neq (c + \mathbf{u})(c + \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$. Etc.

18. No. For example, (1) fails since $2\mathbf{u} - \mathbf{v} \neq 2\mathbf{v} - \mathbf{u}$.

19. Let $\mathbf{0}_1$ and $\mathbf{0}_2$ be zero vectors. Then $\mathbf{0}_1 \oplus \mathbf{0}_2 = \mathbf{0}_1$ and $\mathbf{0}_1 \oplus \mathbf{0}_2 = \mathbf{0}_2$. So $\mathbf{0}_1 = \mathbf{0}_2$.

20. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be negatives of $\mathbf{u}$. Then $\mathbf{u} \oplus \mathbf{u}_1 = \mathbf{0}$ and $\mathbf{u} \oplus \mathbf{u}_2 = \mathbf{0}$. So $\mathbf{u} \oplus \mathbf{u}_1 = \mathbf{u} \oplus \mathbf{u}_2$. Then

$$\begin{aligned} \mathbf{u}_1 \oplus (\mathbf{u} \oplus \mathbf{u}_1) &= \mathbf{u}_1 \oplus (\mathbf{u} \oplus \mathbf{u}_2) \\ (\mathbf{u}_1 \oplus \mathbf{u}) \oplus \mathbf{u}_1 &= (\mathbf{u}_1 \oplus \mathbf{u}) \oplus \mathbf{u}_2 \\ \mathbf{0} \oplus \mathbf{u}_1 &= \mathbf{0} \oplus \mathbf{u}_2 \\ \mathbf{u}_1 &= \mathbf{u}_2. \end{aligned}$$

21. (b) $c \odot \mathbf{0} = c \odot (\mathbf{0} \oplus \mathbf{0}) = c \odot \mathbf{0} \oplus c \odot \mathbf{0}$ so $c \odot \mathbf{0} = \mathbf{0}$.
    (c) Let $c \odot \mathbf{u} = \mathbf{0}$. If $c \neq 0$, then $\frac{1}{c} \odot (c \odot \mathbf{u}) = \frac{1}{c} \odot \mathbf{0} = \mathbf{0}$. Now $\frac{1}{c} \odot (c \odot \mathbf{u}) = \left[\left(\frac{1}{c}\right)(c)\right] \odot \mathbf{u} = 1 \odot \mathbf{u} = \mathbf{u}$, so $\mathbf{u} = \mathbf{0}$.

22. Verify as for Exercise 9. Also, each continuous function is a real valued function.

23. $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}$, so $-(-\mathbf{v}) = \mathbf{v}$.

24. If $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} \oplus \mathbf{w}$, add $-\mathbf{u}$ to both sides.

25. If $a \odot \mathbf{u} = b \odot \mathbf{u}$, then $(a - b) \odot \mathbf{u} = \mathbf{0}$. Now use (c) of Theorem 4.2.

Section 4.3, p. 205

2. Yes.

4. No.

6. (a) and (c).

8. (a).

10. (c).

12. (a) Let

$$A = \begin{bmatrix} a_1 & 0 & b_1 \\ 0 & c_1 & 0 \\ d_1 & 0 & e_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_2 & 0 & b_2 \\ 0 & c_2 & 0 \\ d_2 & 0 & e_2 \end{bmatrix}$$

be any vectors in W. Then

$$A + B = \begin{bmatrix} a_1 + a_2 & 0 & b_1 + b_2 \\ 0 & c_1 + c_2 & 0 \\ d_1 + d_2 & 0 & e_1 + e_2 \end{bmatrix}$$

is in W. Moreover, if k is a scalar, then

$$kA = \begin{bmatrix} ka_1 & 0 & kb_1 \\ 0 & kc_1 & 0 \\ kd_1 & 0 & ke_1 \end{bmatrix}$$

is in W. Hence, W is a subspace of M_{33}.

Alternate solution: Observe that every vector in W can be written as

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so W consists of all linear combinations of five fixed vectors in M_{33}. Hence, W is a subspace of M_{33}.

14. We have

$$A\mathbf{z} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix},$$

so A is in W if and only if $a+b=0$ and $c+d=0$. Thus, W consists of all matrices of the form

$$\begin{bmatrix} a & -a \\ c & -c \end{bmatrix}.$$

Now if

$$A_1 = \begin{bmatrix} a_1 & -a_1 \\ c_1 & -c_1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} a_2 & -a_2 \\ c_2 & -c_2 \end{bmatrix}$$

are in W, then

$$A_1 + A_2 = \begin{bmatrix} a_1 & -a_1 \\ c_1 & -c_1 \end{bmatrix} + \begin{bmatrix} a_2 & -a_2 \\ c_2 & -c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & -(a_1+a_2) \\ c_1+c_2 & -(c_1+c_2) \end{bmatrix}$$

is in W. Moreover, if k is a scalar, then

$$kA_1 = k\begin{bmatrix} a_1 & -a_1 \\ c_1 & -c_1 \end{bmatrix} = \begin{bmatrix} ka_1 & -(ka_1) \\ kc_1 & -(kc_1) \end{bmatrix}$$

is in W. Alternatively, we can observe that every vector in W can be written as

$$\begin{bmatrix} a & -a \\ c & -c \end{bmatrix} = a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix},$$

so W consists of all linear combinations of two fixed vectors in M_{22}. Hence, W is a subspace of M_{22}.

16. (a) and (b).

18. (b) and (c).

20. (a), (b), (c), and (d).

21. Use Theorem 4.3.

22. Use Theorem 4.3.

23. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to $A\mathbf{x} = \mathbf{b}$. Then $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b} + \mathbf{b} \neq \mathbf{b}$ if $\mathbf{b} \neq \mathbf{0}$.

24. $\{\mathbf{0}\}$.

25. Since

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & 6 & 4 \end{bmatrix}\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

it follows that $\begin{bmatrix} t \\ -t \\ t \end{bmatrix}$ is in the null space of A.

26. We have $c\mathbf{x}_0 + d\mathbf{x}_0 = (c+d)\mathbf{x}_0$ is in W, and if r is a scalar then $r(c\mathbf{x}_0) = (rc)\mathbf{x}_0$ is in W.

27. No, it is not a subspace. Let $\mathbf{x}$ be in W so $A\mathbf{x} \neq \mathbf{0}$. Letting $\mathbf{y} = -\mathbf{x}$, we have $\mathbf{y}$ is also in W and $A\mathbf{y} \neq \mathbf{0}$. However, $A(\mathbf{x}+\mathbf{y}) = \mathbf{0}$, so $\mathbf{x}+\mathbf{y}$ does not belong to W.

28. Let V be a subspace of R^1 which is not the zero subspace and let $\mathbf{v} \neq \mathbf{0}$ be any vector in V. If $\mathbf{u}$ is any nonzero vector in R^1, then $\mathbf{u} = \left[\frac{\mathbf{u}}{\mathbf{v}}\right]\mathbf{v}$, so R^1 is a subset of V. Hence, $V = R^1$.

29. Certainly $\{\mathbf{0}\}$ and R^2 are subspaces of R^2. If $\mathbf{u}$ is any nonzero vector then span $\{\mathbf{u}\}$ is a subspace of R^2. To show this, observe that span $\{\mathbf{u}\}$ consists of all vectors in R^2 that are scalar multiples of $\mathbf{u}$. Let $\mathbf{v} = c\mathbf{u}$ and $\mathbf{w} = d\mathbf{u}$ be in span $\{\mathbf{u}\}$ where c and d are any real numbers. Then $\mathbf{v}+\mathbf{w} = c\mathbf{u}+d\mathbf{u} = (c+d)\mathbf{u}$ is in span $\{\mathbf{u}\}$ and if k is any real number, then $k\mathbf{v} = k(c\mathbf{u}) = (kc)\mathbf{u}$ is in span $\{\mathbf{u}\}$. Then by Theorem 4.3, span $\{\mathbf{u}\}$ is a subspace of R^2.

    To show that these are the only subspaces of R^2 we proceed as follows. Let W be any subspace of R^2. Since W is a vector space in its own right, it contains the zero vector $\mathbf{0}$. If $W \neq \{\mathbf{0}\}$, then W contains a nonzero vector $\mathbf{u}$. But then by property (b) of Definition 4.4, W must contain every scalar multiple of $\mathbf{u}$. If every vector in W is a scalar multiple of $\mathbf{u}$ then W is span $\{\mathbf{u}\}$. Otherwise, W contains span $\{\mathbf{u}\}$ and another vector which is not a multiple of $\mathbf{u}$. Call this other vector $\mathbf{v}$. It follows that W contains span $\{\mathbf{u},\mathbf{v}\}$. But in fact span $\{\mathbf{u},\mathbf{v}\} = R^2$. To show this, let $\mathbf{y}$ be any vector in R^2 and let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

    We must show there are scalars c_1 and c_2 such that $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{y}$. This equation leads to the linear system

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

    Consider the transpose of the coefficient matrix:

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}^T = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}.$$

    This matrix is row equivalent to I_2 since its rows are not multiples of each other. Therefore the matrix is nonsingular. It follows that the coefficient matrix is nonsingular and hence the linear system has a solution. Therefore span $\{\mathbf{u},\mathbf{v}\} = R^2$, as required, and hence the only subspaces of R^2 are $\{\mathbf{0}\}$, R^2, or scalar multiples of a single nonzero vector.

30. (b) Use Exercise 25. The depicted set represents all scalar multiples of a nonzero vector, hence is a subspace.

31. We have

$$\begin{bmatrix} a & b & c \\ a & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b & a+b \\ a & 0 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = a\mathbf{w}_1 + b\mathbf{w}_2.$$

32. Every vector in W is of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$, which can be written as

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3,$$

    where

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

34. (a) and (c).

35. (a) The line l_0 consists of all vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Use Theorem 4.3.

(b) The line l through the point $P_0(x_0, y_0, z_0)$ consists of all vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

If P_0 is not the origin, the conditions of Theorem 4.3 are not satisfied.

36. (d)

38. (a) $x = 3 + 4t$, $y = 4 - 5t$, $z = -2 + 2t$. (b) $x = 3 - 2t$, $y = 2 + 5t$, $z = 4 + t$.

42. Use matrix multiplication $\mathbf{c}A$ where $\mathbf{c}$ is a row vector containing the coefficients and matrix A has rows that are the vectors from R_n.

Section 4.4, p. 215

2. (a) 1 does not belong to span S.

(b) Span S consists of all vectors of the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$, where a is any real number. Thus, the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not in span S.

(c) Span S consists of all vectors of M_{22} of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, where a and b are any real numbers. Thus, the vector $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not in span S.

4. (a) Yes. (b) Yes. (c) No. (d) No.

6. (d).

8. (a) and (c).

10. Yes.

12. $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

13. Every vector A in W is of the form $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, where a, b, and c are any real numbers. We have

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

so A is in span S. Thus, every vector in W is in span S. Hence, span $S = W$.

14. $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \right.$
$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$

16. From Exercise 43 in Section 1.3, we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, and $\mathrm{Tr}(AB-BA) = \mathrm{Tr}(AB)-\mathrm{Tr}(BA) = 0$. Hence, span T is a subset of the set S of all $n \times n$ matrices with trace $= 0$. However, S is a proper subset of M_{nn}.

Section 4.5, p. 226

1. We form Equation (1):
$$c_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$
which has nontrivial solutions. Hence, S is linearly dependent.

2. We form Equation (1):
$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$
which has only the trivial solution. Hence, S is linearly independent.

4. No.

6. Linearly dependent.

8. Linearly independent.

10. Yes.

12. (b) and (c) are linearly independent, (a) is linearly dependent.
$\begin{bmatrix} 4 & 6 \\ 8 & 6 \end{bmatrix} = 3 \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + 1 \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}.$

14. Only (d) is linearly dependent: $\cos 2t = \cos^2 t - \sin^2 t$.

16. $c = 1$.

18. Suppose that $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent. Then $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$, where c_1 and c_2 are not both zero. Say $c_2 \neq 0$. Then $\mathbf{v} = -\left(\frac{c_1}{c_2}\right)\mathbf{u}$. Conversely, if $\mathbf{v} = k\mathbf{u}$, then $k\mathbf{u} - 1\mathbf{v} = \mathbf{0}$. Since the coefficient of $\mathbf{v}$ is nonzero, $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent.

19. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be linearly dependent. Then $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$, where at least one of the coefficients $a_1, a_2, \dots, a_k$ is not zero. Say that $a_j \neq 0$. Then
$$\mathbf{v}_j = -\frac{a_1}{a_j}\mathbf{v}_1 - \frac{a_2}{a_j}\mathbf{v}_2 - \cdots - \frac{a_{j-1}}{a_j}\mathbf{v}_{j-1} - \frac{a_{j+1}}{a_j}\mathbf{v}_{j+1} - \cdots - \frac{a_k}{a_j}\mathbf{v}_k.$$

20. Suppose $a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = a_1(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + a_2(\mathbf{v}_2 + \mathbf{v}_3) + a_3\mathbf{v}_3 = \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, $a_1 = 0$, $a_1 + a_2 = 0$ (and hence $a_2 = 0$), and $a_1 + a_2 + a_3 = 0$ (and hence $a_3 = 0$). Thus $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

21. Form the linear combination
$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$$
which gives
$$c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) = (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3 = \mathbf{0}.$$
Since S is linearly independent we have
$$\begin{aligned} c_1 + c_2 \phantom{{}+c_3} &= 0 \\ c_1 \phantom{{}+c_2} + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$
a linear system whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$. The reduced row echelon form is
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$
thus $c_1 = c_2 = c_3 = 0$ which implies that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

22. Form the linear combination
$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$$
which gives
$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = (c_1 + c_2 + c_3)\mathbf{v}_1 + (c_2 + c_3)\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$
Since S is linearly dependent, this last equation is satisfied with $c_1 + c_2 + c_3$, c_3, and $c_2 + c_3$ not all being zero. This implies that c_1, c_2, and c_3 are not all zero. Hence, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly dependent.

23. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Then one of the $\mathbf{v}_j$'s is a linear combination of the preceding vectors in the list. It must be $\mathbf{v}_3$ since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Thus $\mathbf{v}_3$ belongs to span $\{\mathbf{v}_1, \mathbf{v}_2\}$. Contradiction.

24. Form the linear combination
$$c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_nA\mathbf{v}_n = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = \mathbf{0}.$$
Since A is nonsingular, Theorem 2.9 implies that
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$
Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent, we have $c_1 = c_2 = \cdots = c_n = 0$. Hence, $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$ is linearly independent.

25. Let A have k nonzero rows, which we denote by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ where
$$\mathbf{v}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & 1 & \cdots & a_{in} \end{bmatrix}.$$
Let $c_1 < c_2 < \cdots < c_k$ be the columns in which the leading entries of the k nonzero rows occur. Thus $\mathbf{v}_i = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 & a_{i\,c_{i+1}} & \cdots & a_{in} \end{bmatrix}$ that is, $a_{ij} = 0$ for $j < c_i$ and $c_{ic_i} = 1$. If $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$, examining the c_1th entry on the left yields $a_1 = 0$, examining the c_2th entry yields $a_2 = 0$, and so forth. Therefore $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent.

26. Let $\mathbf{v}_j = \sum_{i=1}^{k} a_{ij}\mathbf{u}_i$. Then $\mathbf{w} = \sum_{j=1}^{m} b_j\mathbf{v}_j = \sum_{j=1}^{m} b_j \sum_{i=1}^{k} a_{ij}\mathbf{u}_i = \sum_{i=1}^{k}\left[\sum_{j=1}^{m} a_{ij}b_j\right]\mathbf{u}_i$.

27. In R^1 let $S_1 = \{1\}$ and $S_2 = \{1, 0\}$. S_1 is linearly independent and S_2 is linearly dependent.

28. See Exercise 27 above.

29. In MATLAB the command **null(A)** produces an orthonormal basis for the null space of A.

31. Each set of two vectors is linearly independent since they are not scalar multiples of one another. In MATLAB the reduced row echelon form command implies sets (a) and (b) are linearly independent while (c) is linearly dependent.

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2. (c).

4. (d).

6. If $c_1\begin{bmatrix}1&1\\0&0\end{bmatrix} + c_2\begin{bmatrix}0&0\\1&1\end{bmatrix} + c_3\begin{bmatrix}1&0\\0&1\end{bmatrix} + c_4\begin{bmatrix}0&1\\1&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$, then $\begin{bmatrix}c_1+c_3 & c_1+c_4\\ c_2+c_4 & c_2+c_3+c_4\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$. The first three entries imply $c_3 = -c_1 = c_4 = -c_2$. The fourth entry gives $c_2 - c_2 - c_2 = -c_2 = 0$. Thus $c_i = 0$ for $i = 1, 2, 3, 4$. Hence the set of four matrices is linearly independent. By Theorem 4.12, it is a basis.

8. (b) is a basis for R^3 and $\begin{bmatrix}2\\1\\3\end{bmatrix} = 1\begin{bmatrix}1\\1\\2\end{bmatrix} + 2\begin{bmatrix}2\\2\\0\end{bmatrix} - 1\begin{bmatrix}3\\4\\-1\end{bmatrix}$.

10. (a) forms a basis: $5t^2 - 3t + 8 = -3(t^2 + t) + 0t^2 + 8(t^2 + 1)$.

12. A possible answer is $\left\{\begin{bmatrix}1&1&0&-1\end{bmatrix}, \begin{bmatrix}0&1&2&1\end{bmatrix}, \begin{bmatrix}0&0&3&1\end{bmatrix}\right\}$; $\dim W = 3$.

14. $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$.

16. $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$.

18. A possible answer is $\{\cos^2 t, \sin^2 t\}$ is a basis for W; $\dim W = 2$.

20. (a) $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, (b) $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\-1\end{bmatrix}\right\}$, (c) $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\0\\1\end{bmatrix}\right\}$.

22. $\{t^3 + 5t^2 + t, 3t^2 - 2t + 1\}$.

24. (a) 3. (b) 2.

26. (a) 2. (b) 3. (c) 3. (d) 3.

28. (a) A possible answer is $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$. (b) A possible answer is $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$.

30. $\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}$;

    $\dim M_{23} = 6$. $\dim M_{mn} = mn$.

32. 2.

34. The set of all polynomials of the form $at^3 + bt^2 + (b - a)$, where a and b are any real numbers.

35. We show that $\{c\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is also a set of $k = \dim V$ vectors which spans V. If $\mathbf{v} = \sum_{i=1}^{n} a_i\mathbf{v}_i$ is a vector in V, then
$$\mathbf{v} = \left[\frac{a_1}{c}\right](c\mathbf{v}_1) + \sum_{i=2}^{n} a_i\mathbf{v}_i.$$

36. Let $d = \max\{d_1, d_2, \ldots, d_k\}$. The polynomial $t^{d+1} + t^d + \cdots + t + 1$ cannot be written as a linear combination of polynomials of degrees $\leq d$.

37. If $\dim V = n$, then V has a basis consisting of n vectors. Theorem 4.10 then implies the result.

38. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a minimal spanning set for V. From Theorem 4.9, S contains a basis T for V. Since T spans S and S is a spanning set for V, $T = S$. It follows from Corollary 4.1 that $k = n$.

39. Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, $m > n$ be a set of vectors in V. Since $m > n$, Theorem 4.10 implies that T is linearly dependent.

40. Let $\dim V = n$ and let S be a set of vectors in V containing m elements, $m < n$. Assume that S spans V. By Theorem 4.9, S contains a basis T for V. Then T must contain n elements. This contradiction implies that S cannot span V.

41. Let $\dim V = n$. First observe that any set of vectors in W that is linearly independent in W is linearly independent in V. If $W = \{\mathbf{0}\}$, then $\dim W = 0$ and we are done. Suppose now that W is a nonzero subspace of V. Then W contains a nonzero vector $\mathbf{v}_1$, so $\{\mathbf{v}_1\}$ is linearly independent in W (and in V). If span $\{\mathbf{v}_1\} = W$, then $\dim W = 1$ and we are done. If span $\{\mathbf{v}_1\} \neq W$, then there exists a vector $\mathbf{v}_2$ in W which is not in span $\{\mathbf{v}_1\}$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent in W (and in V). Since $\dim V = n$, no linearly independent set of vectors in V can have more than n vectors. Hence, no linearly independent set of vectors in W can have more than n vectors. Continuing the above process we find a basis for W containing at most n vectors. Hence $\dim W \leq \dim V$.

42. Let $\dim V = \dim W = n$. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for W. Then S is also a basis for V, by Theorem 4.13. Hence, $V = W$.

43. Let $V = R^3$. The trivial subspaces of any vector space are $\{\mathbf{0}\}$ and V. Hence $\{\mathbf{0}\}$ and R^3 are subspaces of R^3. In Exercise 35 in Section 4.3 we showed that any line ℓ through the origin is a subspace of R^3. Thus we need only show that any plane π passing through the origin is a subspace of R^3. Any plane π in R^3 through the origin has an equation of the form $ax + by + cz = 0$. Sums and scalar multiples of any point on π will also satisfy this equation, hence π is a subspace of R^3. To show that $\{\mathbf{0}\}$, V, lines, and planes through the origin are the only subspaces of R^3 we argue in a manner similar to that given in Exercise 29 in Section 4.3 which considered a similar problem in R^2. Let W be any subspace of R^3. Hence W contains the zero vector $\mathbf{0}$. If $W \neq \{\mathbf{0}\}$ then it contains a nonzero vector $\mathbf{v} = \begin{bmatrix}a & b & c\end{bmatrix}^T$ where at least one of a, b, or c is not zero. Since W is a subspace it contains span $\{\mathbf{v}\}$. If $W =$ span $\{\mathbf{v}\}$ then W is a line in R^3 through the origin. Otherwise, there exists a vector $\mathbf{u}$ in W which is not in span $\{\mathbf{v}\}$. Hence $\{\mathbf{v}, \mathbf{u}\}$ is a linearly independent set. But then W contains span $\{\mathbf{v}, \mathbf{u}\}$. If $W =$ span $\{\mathbf{v}, \mathbf{u}\}$ then W is a plane through the origin. Otherwise there is a vector $\mathbf{x}$ in W that is not in span $\{\mathbf{v}, \mathbf{u}\}$. Hence $\{\mathbf{v}, \mathbf{u}, \mathbf{x}\}$ is a linearly independent set in W and W contains span $\{\mathbf{v}, \mathbf{u}, \mathbf{x}\}$. But $\{\mathbf{v}, \mathbf{u}, \mathbf{x}\}$ is a maximal linearly independent set in R^3, hence a basis for R^3. It follows in this case that $W = R^3$.

44. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Since every vector in V can be written as a linear combination of the vectors in S, it follows that S spans V. Suppose now that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}.$$

We also have

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

From the hypothesis it then follows that $a_1 = 0$, $a_2 = 0$, $\ldots$, $a_n = 0$. Hence, S is a basis for V.

45. (a) If span $S \neq V$, then there exists a vector $\mathbf{v}$ in V that is not in S. Vector $\mathbf{v}$ cannot be the zero vector since the zero vector is in every subspace and hence in span S. Hence $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{v}\}$ is a linearly independent set. This follows since $\mathbf{v}_i$, $i = 1, \ldots, n$ are linearly independent and $\mathbf{v}$ is not a linear combination of the $\mathbf{v}_i$. But this contradicts Corollary 4.4. Hence our assumption that span $S \neq V$ is incorrect. Thus span $S = V$. Since S is linearly independent and spans V it is a basis for V.

    (b) We want to show that S is linearly independent. Suppose S is linearly dependent. Then there is a subset of S consisting of at most $n-1$ vectors which is a basis for V. (This follows from Theorem 4.9) But this contradicts $\dim V = n$. Hence our assumption is false and S is linearly independent. Since S spans V and is linearly independent it is a basis for V.

46. Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a maximal independent subset of S, and let $\mathbf{v}$ be any vector in S. Since T is a maximal independent subset then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}\}$ is linearly dependent, and from Theorem 4.7 it follows that $\mathbf{v}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, that is, of the vectors in T. Since S spans V, we find that T also spans V and is thus a basis for V.

47. If A is nonsingular then the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. Let

$$c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_nA\mathbf{v}_n = \mathbf{0}.$$

Then $A(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = \mathbf{0}$ and by the opening remark we must have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

However since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent it follows that $c_1 = c_2 = \cdots = c_n = 0$. Hence $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$ is linearly independent.

48. Since A is singular, Theorem 2.9 implies that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in R^n, it is a basis for R^n, so

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Observe that $\mathbf{x} \neq \mathbf{0}$, so $c_1, c_2, \ldots, c_n$ are not all zero. Then

$$\mathbf{0} = A\mathbf{x} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \cdots + c_n(A\mathbf{v}_n).$$

Hence, $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$ is linearly dependent.

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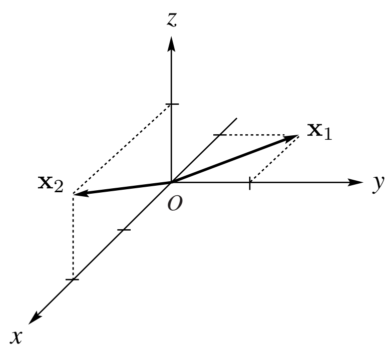
2. (a) $x = -r + 2s$, $y = r$, $z = s$, where r, s are any real numbers.

   (b) Let $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Then

$$\begin{bmatrix} -r + 2s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = r\mathbf{x}_1 + s\mathbf{x}_2.$$

(c)



4. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$; dimension $= 3$.

6. $\left\{ \begin{bmatrix} 4 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$; dimension $= 2$.

8. No basis; dimension $= 0$.

10. $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 17 \\ 0 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$; dimension $= 2$.

12. $\left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

14. $\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$.

16. No basis.

18. $\lambda = 3, -2$.

20. $\lambda = 1, 2, -2$

22. $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{x}_h = r \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, r any number.

23. Since each vector in S is a solution to $A\mathbf{x} = \mathbf{0}$, we have $A\mathbf{x}_i = \mathbf{0}$ for $i = 1, 2, \ldots, n$. The span of S consists of all possible linear combinations of the vectors in S. Hence

$$\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

represents an arbitrary member of span S. We have

$$A\mathbf{y} = c_1 A\mathbf{x}_1 + c_2 A\mathbf{x}_2 + \cdots + c_k A\mathbf{x}_k = c_1\mathbf{0} + c_2\mathbf{0} + \cdots + c_k\mathbf{0} = \mathbf{0}.$$

24. If A has a row or column of zeros, then A is singular (Exercise 46 in Section 1.5), so by Theorem 2.9, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

25. (a) Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. Since the dimension of the null space of A is 3, the null space of A is R^3. Then the natural basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for the null space of A. Forming $A\mathbf{e}_1 = \mathbf{0}$, $A\mathbf{e}_2 = \mathbf{0}$, $A\mathbf{e}_3 = \mathbf{0}$, we find that all the columns of A must be zero. Hence, $A = O$.

    (b) Since $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, the null space of A contains a nonzero vector, so the dimension of the null space of A is not zero. If this dimension is 3, then by part (a), $A = O$, a contradiction. Hence, the dimension is either 1 or 2.

26. Since the reduced row echelon forms of matrices A and B are the same it follows that the solutions to the linear systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are the same set of vectors. Hence the null spaces of A and B are the same.

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2. $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

4. $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

6. $\begin{bmatrix} -1 \\ 2 \\ -2 \\ 4 \end{bmatrix}$.

8. $(3, 1, 3)$.

10. $t^2 - 3t + 2$.

12. $\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$.

13. (a) To show S is a basis for R^2 we show that the set is linearly independent and since $\dim R^2 = 2$ we can conclude they are a basis. The linear combination

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

leads to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -2 & 0 \end{array}\right].$$

The reduced row echelon form of this homogeneous system is $\left[\begin{array}{c|c} I_2 & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right]$ so the set S is linearly independent.

(b) Find c_1 and c_2 so that

$$c_1\begin{bmatrix}1\\-1\end{bmatrix}+c_2\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}.$$

The corresponding linear system has augmented matrix

$$\left[\begin{array}{rr|r}1&1&2\\-1&-2&6\end{array}\right].$$

The reduced row echelon form is $\left[\begin{array}{rr|r}1&0&10\\0&1&-8\end{array}\right]$ so $\left[\mathbf{v}\right]_S=\begin{bmatrix}10\\-8\end{bmatrix}$.

(c) $A\mathbf{v}_1=\begin{bmatrix}-0.3\\0.3\end{bmatrix}=-.3\begin{bmatrix}1\\-1\end{bmatrix}=-0.3\mathbf{v}_1$, so $\lambda_1=-0.3$.

(d) $A\mathbf{v}_2=\begin{bmatrix}0.25\\-.50\end{bmatrix}=0.25\begin{bmatrix}1\\-2\end{bmatrix}=0.25\mathbf{v}_2$ so $\lambda_2=0.25$.

(e) $\mathbf{v}=10\mathbf{v}_1-8\mathbf{v}_2$ so $A^n\mathbf{v}=10A^n\mathbf{v}_1-8A^n\mathbf{v}_2=10(\lambda_1)^n\mathbf{v}_1-8(\lambda_2)^n\mathbf{v}_2$.

(f) As n increases, the limit of the sequence is the zero vector.

14. (a) Since $\dim R^2=2$, we show that S is a linearly independent set. The augmented matrix corresponding to

$$c_1\begin{bmatrix}1\\-1\end{bmatrix}+c_2\begin{bmatrix}-2\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

is $\left[\begin{array}{rr|r}1&-2&0\\-1&3&0\end{array}\right]$. The reduced row echelon form is $\left[\begin{array}{c|c}I_2&0\end{array}\right]$ so S is a linearly independent set.

(b) Set

$$\mathbf{v}=\begin{bmatrix}4\\3\end{bmatrix}=c_1\begin{bmatrix}1\\-1\end{bmatrix}+c_2\begin{bmatrix}-2\\3\end{bmatrix}.$$

Solving for c_1 and c_2 we find $c_1=18$ and $c_2=7$. Thus $\left[\mathbf{v}\right]_S=\begin{bmatrix}18\\7\end{bmatrix}$.

(c) $A\mathbf{v}_1=\begin{bmatrix}-1&-2\\3&4\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$, so $\lambda_1=1$.

(d) $A\mathbf{v}_2=\begin{bmatrix}-1&-2\\3&4\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}=\begin{bmatrix}-4\\6\end{bmatrix}=2\begin{bmatrix}-2\\3\end{bmatrix}$, so $\lambda_2=2$.

(e) $A^n\mathbf{v}=A^n[18\mathbf{v}_1+7\mathbf{v}_2]=18A^n\mathbf{v}_1+7A^n\mathbf{v}_2=18(1)^n\mathbf{v}_1+7(2)^n\mathbf{v}_2=18\mathbf{v}_1+7(2^n)\mathbf{v}_2$.

(f) As n increases the sequence becomes unbounded since $\lim_{n\to\infty}A^n\mathbf{v}=18\mathbf{v}_1+7\mathbf{v}_2\lim_{n\to\infty}2^n$.

16. (a) $\left[\mathbf{v}\right]_T=\begin{bmatrix}-9\\-8\\28\end{bmatrix}$, $\left[\mathbf{w}\right]_T=\begin{bmatrix}1\\-2\\13\end{bmatrix}$

(b) $P_{S\leftarrow T}=\begin{bmatrix}-2&-5&-2\\-1&-6&-2\\1&2&1\end{bmatrix}$.

(c) $\left[\mathbf{v}\right]_S=\begin{bmatrix}2\\1\\3\end{bmatrix}$, $\left[\mathbf{w}\right]_S=\begin{bmatrix}-18\\-17\\8\end{bmatrix}$.

(d) Same as (c).

(e) $Q_{T\leftarrow S}=\begin{bmatrix}-2&1&-2\\-1&0&-2\\4&-1&7\end{bmatrix}$.

(f) Same as (a).

18. (a) $\left[\mathbf{v}\right]_T=\begin{bmatrix}4\\-2\\1\end{bmatrix}$, $\left[\mathbf{w}\right]_T=\begin{bmatrix}0\\8\\-6\end{bmatrix}$.

(b) $P_{S\leftarrow T}=\begin{bmatrix}0&1&0\\\frac{1}{2}&-\frac{1}{2}&0\\-\frac{1}{2}&\frac{1}{2}&1\end{bmatrix}$.

(c) $\left[\mathbf{v}\right]_S = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$, $\left[\mathbf{w}\right]_S = \begin{bmatrix} 8 \\ -4 \\ -2 \end{bmatrix}$. (d) Same as (c).

(e) $Q_{T \leftarrow S} = r \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. (f) Same as (a).

20. $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

22. $\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$.

24. $T = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$.

26. $T = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

28. (a) V is isomorphic to itself. Let $L\colon V \to V$ be defined by $L(\mathbf{v}) = \mathbf{v}$ for $\mathbf{v}$ in V; that is, L is the identity map.
(b) If V is isomorphic to W, then there is an isomorphism $L\colon V \to W$ which is a one-to-one and onto mapping. Then $L^{-1}\colon W \to V$ exists. Verify that L^{-1} is one-to-one and onto and is also an isomorphism. This is all done in the proof of Theorem 6.7.
(c) If U is isomorphic to V, let $L_1\colon U \to V$ be an isomorphism. If V is isomorphic to W, let $L_2\colon V \to W$ be an isomorphism. Let $L\colon U \to W$ be defined by $L(\mathbf{v}) = L_2(L_1(\mathbf{v}))$ for $\mathbf{v}$ in U. Verify that L is an isomorphism.

29. (a) $L(\mathbf{0}_V) = L(\mathbf{0}_V + \mathbf{0}_V) = L(\mathbf{0}_V) + L(\mathbf{0}_V)$, so $L(\mathbf{0}_V) = \mathbf{0}_W$.
(b) $L(\mathbf{v} - \mathbf{w}) = L(\mathbf{v} + (-1)\mathbf{w}) = L(\mathbf{v}) + L((-1)\mathbf{w}) = L(\mathbf{v}) + (-1)L(\mathbf{w}) = L(\mathbf{v}) - L(\mathbf{w})$.

30. By Theorem 3.15, R^n and R^m are isomorphic if and only if their dimensions are equal.

31. Let $L\colon R_n \to R^n$ be defined by

$$L\left(\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Verify that L is an isomorphism.

32. Let $L\colon P_2 \to R^3$ be defined by $L(at^2 + bt + c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Verify that L is an isomorphism.

33. (a) Let $L\colon M_{22} \to R^4$ be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Verify that L is an isomorphism.

(b) $\dim M_{22} = 4$.

34. If $\mathbf{v}$ is any vector in V, then $\mathbf{v} = ae^t + be^{-t}$, where a and b are scalars. Then let $L: V \to R^2$ be defined by $L(\mathbf{v}) = \begin{bmatrix} a \\ b \end{bmatrix}$. Verify that L is an isomorphism.

35. From Exercise 18 in Section 4.6, $V = \operatorname{span} S$ has a basis $\{\sin^2 t, \cos^2 t\}$ hence $\dim V = 2$. It follows from Theorem 4.14 that V is isomorphic to R_2.

36. Let V and W be isomorphic under the isomorphism L. If V_1 is a subspace of V then $W_1 = L(V)$ is a subspace of W which is isomorphic to V_1.

37. Let $\mathbf{v} = \mathbf{w}$. The coordinates of a vector relative to basis S are the coefficients used to express the vector in terms of the members of S. A vector has a unique expression in terms of the vectors of a basis, hence it follows that $\left[\mathbf{v}\right]_S$ must equal $\left[\mathbf{w}\right]_S$. Conversely, let

$$\left[\mathbf{v}\right]_S = \left[\mathbf{w}\right]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

then $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ and $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$. Hence $\mathbf{v} = \mathbf{w}$.

38. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$, $\mathbf{w} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$. Then

$$\left[\mathbf{v}\right]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \left[\mathbf{w}\right]_S = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

We also have

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2 + \cdots + (a_n + b_n)\mathbf{v}_n \\ c\mathbf{v} &= (ca_1)\mathbf{v}_1 + (ca_2)\mathbf{v}_2 + \cdots + (ca_n)\mathbf{v}_n, \end{aligned}$$

so

$$\left[\mathbf{v} + \mathbf{w}\right]_S = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \left[\mathbf{v}\right]_S + \left[\mathbf{w}\right]_S$$

$$\left[c\mathbf{v}\right]_S = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} = c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = c\left[\mathbf{v}\right]_S.$$

39. Consider the homogeneous system $M_S\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$. This system can then be written in terms of the columns of M_S as

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0},$$

where $\mathbf{v}_j$ is the jth column of M_S. Since $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent, we have $a_1 = a_2 = \cdots = a_n = 0$. Thus, $\mathbf{x} = \mathbf{0}$ is the only solution to $M_S\mathbf{x} = \mathbf{0}$, so by Theorem 2.9 we conclude that M_S is nonsingular.

40. Let $\mathbf{v}$ be a vector in V. Then $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$. This last equation can be written in matrix form as

$$\mathbf{v} = M_S \left[\mathbf{v}\right]_S$$

where M_S is the matrix whose jth column is $\mathbf{v}_j$. Similarly, $\mathbf{v} = M_T \left[\mathbf{v}\right]_T$.

41. (a) From Exercise 40 we have

$$M_S \left[\mathbf{v}\right]_S = M_T \left[\mathbf{v}\right]_T .$$

From Exercise 39 we know that M_S is nonsingular, so

$$\left[\mathbf{v}\right]_S = M_S^{-1} M_T \left[\mathbf{v}\right]_T .$$

Equation (3) is

$$\left[\mathbf{v}\right]_S = P_{S\leftarrow T} \left[\mathbf{v}\right]_T ,$$

so

$$P_{S\leftarrow T} = M_S^{-1} M_T.$$

(b) Since M_S and M_T are nonsingular, M_S^{-1} is nonsingular, so $P_{S\leftarrow T}$, as the product of two nonsingular matrices, is nonsingular.

(c) $M_S = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $M_T = \begin{bmatrix} 6 & 4 & 5 \\ 3 & -1 & 5 \\ 3 & 3 & 2 \end{bmatrix}$, $M_S^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \end{bmatrix}$, $P_{S\leftarrow T} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

42. Suppose that $\left\{ \left[\mathbf{w}_1\right]_S, \left[\mathbf{w}_2\right]_S, \ldots, \left[\mathbf{w}_k\right]_S \right\}$ is linearly dependent. Then there exist scalars, a_i, $i = 1, 2, \ldots, k$, not all zero such that

$$a_1 \left[\mathbf{w}_1\right]_S + a_2 \left[\mathbf{w}_2\right]_S + \cdots + a_k \left[\mathbf{w}_k\right]_S = \left[\mathbf{0}_V\right]_S .$$

Using Exercise 38 we find that the preceding equation is equivalent to

$$\left[a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k\right]_S = \left[\mathbf{0}_V\right]_S .$$

By Exercise 37 we have

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k = \mathbf{0}_V .$$

Since the $\mathbf{w}$'s are linearly independent, the preceding equation is only true when all $a_i = 0$. Hence we have a contradiction and our assumption that the $\left[\mathbf{w}_i\right]$'s are linearly dependent must be false. It follows that $\left\{ \left[\mathbf{w}_1\right]_S, \left[\mathbf{w}_2\right]_S, \ldots, \left[\mathbf{w}_k\right]_S \right\}$ is linearly independent.

43. From Exercise 42 we know that $T = \left\{ \left[\mathbf{v}_1\right]_S, \left[\mathbf{v}_2\right]_S, \ldots, \left[\mathbf{v}_n\right]_S \right\}$ is a linearly independent set of vectors in R^n. By Theorem 4.12, T spans R^n and is thus a basis for R^n.

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2. A possible answer is $\{t^3, t^2, t, 1\}$.

4. A possible answer is $\left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right\}$.

6. (a) $\left\{ \left(1, 0, 0, -\frac{33}{7}\right), \left(0, 1, 0, \frac{23}{7}\right), \left(0, 0, 1, -\frac{8}{7}\right) \right\}$. (b) $\{(1, 2, -1, 3), (3, 5, 2, 0), (0, 1, 2, 1)\}$.

8. (a) $\left\{\begin{bmatrix}1\\0\\4\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-\frac{5}{2}\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$. (b) $\left\{\begin{bmatrix}-2\\-2\\-3\\4\end{bmatrix}, \begin{bmatrix}2\\2\\3\\-2\end{bmatrix}, \begin{bmatrix}3\\4\\2\\1\end{bmatrix}\right\}$.

10. (a) 2. (b) 2.

11. The result follows from the observation that the nonzero rows of A are linearly independent and span the row space of A.

12. (a) 3. (b) 2. (c) 2.

14. (a) rank $= 2$, nullity $= 2$. (b) rank $= 4$, nullity $= 0$.

16. (a) and (b) are consistent.

18. (b).

20. (a).

22. (a).

24. (a) 3. (b) 3.

26. No.

28. Yes, linearly independent.

30. Yes.

32. Yes.

34. (a) 3.
    (b) The six columns of A span a column space of dimension rank A, which is at most 4. Thus the six columns are linearly dependent.
    (c) The five rows of A span a row space of dimension rank A, which is at most 3. Thus the five rows are linearly dependent.

36. (a) 0, 1, 2, 3. (b) 3. (c) 2.

37. S is linearly independent if and only if the n rows of A are linearly independent if and only if rank A $= n$.

38. S is linearly independent if and only if the column rank of $A = n$ if and only if rank $A = n$.

39. If $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution then A is singular, rank $A < n$, and the columns of A are linearly dependent, and conversely.

40. If rank $A = n$, then the dimension of the column space of A is n. Since the columns of A span its column space, it follows by Theorem 4.12 that they form a basis for the column space and are thus linearly independent. Conversely, if the columns of A are linearly independent, then the dimension of the column space is n, so rank $A = n$.

41. If the rows of A are linearly independent, then rank $A = n$ and the columns of A span R^n.

42. From the definition of reduced row echelon form, any column in which a leading one appears must be a column of an identity matrix. Assuming that $\mathbf{v}_i$ has its first nonzero entry in position j_i, for $i = 1, 2, \ldots, k$, every other vector in S must have a zero in position j_i. Hence if $\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k$, it follows that $a_{j_i} = b_i$ as desired.

43. Let rank $A = n$. Then Corollary 4.7 implies that A is nonsingular, so $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then $A\mathbf{x}_1 = A\mathbf{x}_2$ and multiplying both sides by A^{-1}, we have $\mathbf{x}_1 = \mathbf{x}_2$. Thus, $A\mathbf{x} = \mathbf{b}$ has a unique solution.

    Conversely, suppose that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$. Then the n linear systems $A\mathbf{x} = \mathbf{e}_1$, $A\mathbf{x} = \mathbf{e}_2$, $\dots$, $A\mathbf{x} = \mathbf{e}_n$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the columns of I_n, have solutions $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Let B be the matrix whose jth column is $\mathbf{x}_j$. Then the n linear systems above can be written as $AB = I_n$. Hence, $B = A^{-1}$, so A is nonsingular and Corollary 4.7 implies that rank $A = n$.

44. Let $A\mathbf{x} = \mathbf{b}$ have a solution for every $m \times 1$ matrix $\mathbf{b}$. Then the columns of A span R^m. Thus there is a subset of m columns of A that is a basis for R^m and rank $A = m$. Conversely, if rank $A = m$, then column rank $A = m$. Thus m columns of A are a basis for R^m and hence all the columns of A span R^m. Since $\mathbf{b}$ is in R^m, it is a linear combination of the columns of A; that is, $A\mathbf{x} = \mathbf{b}$ has a solution for every $m \times 1$ matrix $\mathbf{b}$.

45. Since the rank of a matrix is the same as its row rank and column rank, the number of linearly independent rows of a matrix is the same as the number of linearly independent columns. It follows that the largest the rank can be is $\min\{m, n\}$. Since $m \neq n$, it must be that either the rows or columns are linearly dependent.

46. Suppose that $A\mathbf{x} = \mathbf{b}$ is consistent. Assume that there are at least two different solutions $\mathbf{x}_1$ and $\mathbf{x}_2$. Then $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$, so $A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$. That is, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution so nullity $A > 0$. By Theorem 4.19, rank $A < n$. Conversely, if rank $A < n$, then by Corollary 4.8, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{y}$. Suppose that $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{b}$. Thus, $A\mathbf{y} = \mathbf{0}$ and $A\mathbf{x}_0 = \mathbf{b}$. Then $\mathbf{x}_0 + \mathbf{y}$ is a solution to $A\mathbf{x} = \mathbf{b}$, since $A(\mathbf{x}_0 + \mathbf{y}) = A\mathbf{x}_0 + A\mathbf{y} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. Since $\mathbf{y} \neq \mathbf{0}$, $\mathbf{x}_0 + \mathbf{y} \neq \mathbf{x}_0$, so $A\mathbf{x} = \mathbf{b}$ has more than one solution.

47. The solution space is a vector space of dimension d, $d \geq 2$.

48. No. If all the nontrivial solutions of the homogeneous system are multiples of each other, then the dimension of the solution space is 1. The rank of the coefficient matrix is ≤ 5. Since nullity $= 7 - \text{rank}$, nullity $\geq 7 - 5 = 2$.

49. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ spans R^n (R_n). Then by Theorem 4.11, S is linearly independent and hence the dimension of the column space of A is n. Thus, rank $A = n$. Conversely, if rank $A = n$, then the set S consisting of the columns (rows) of A is linearly independent. By Theorem 4.12, S spans R^n.

Supplementary Exercises for Chapter 4, p. 285

1. (a) The verification of Definition 4.4 follows from the properties of continuous functions and real numbers. In particular, in calculus it is shown that the sum of continuous functions is continuous and that a real number times a continuous function is again a continuous function. This verifies (a) and (b) of Definition 4.4. We demonstrate that (1) and (5) hold and (2), (3), (4), (6), (7), (8) are shown in a similar way. To show (1), let f and g belong to $C[a, b]$ and for t in $[a, b]$

$$(f \oplus g)(t) = f(t) + g(t) = g(t) + f(t) = (g \oplus f)(t)$$

since $f(t)$ and $g(t)$ are real numbers and the addition of real numbers is commutative. To show (5), let c be any real number. Then

$$\begin{aligned} c \odot (f \oplus g)(t) &= c(f(t) + g(t)) = cf(t) + cg(t) \\ &= c \odot f(t) + c \odot g(t) = (c \odot f \oplus c \odot g)(t) \end{aligned}$$

since c, $f(t)$, and $g(t)$ are real numbers and multiplication of real numbers distributes over addition or real numbers.

(b) $k = 0$.

(c) Let f and g have roots at t_i, $i = 1, 2, \ldots, n$; that is, $f(t_i) = g(t_i) = 0$. It follows that $f \oplus g$ has roots at t_i, since $(f \oplus g)(t_i) = f(t_i) + g(t_i) = 0 + 0 = 0$. Similarly, $k \odot f$ has roots at t_i since $(k \odot f)(t_i) = kf(t_i) = k \cdot 0 = 0$.

2. (a) Let $\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ be in W. Then $a_4 - a_3 = a_2 - a_1$ and $b_4 - b_3 = b_2 - b_1$. It follows that

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ a_4 + b_4 \end{bmatrix}$$

and

$$(a_4 + b_4) - (a_3 + b_3) = (a_4 - a_3) + (b_4 - b_3) = (a_2 - a_1) + (b_2 - b_1) = (a_2 + b_2) - (a_1 + b_1),$$

so $\mathbf{v} + \mathbf{w}$ is in W. Similarly, if c is any real number,

$$c\mathbf{v} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \\ ca_4 \end{bmatrix}$$

and

$$ca_4 - ca_3 = c(a_4 - a_3) = c(a_2 - a_1) = ca_2 - ca_1$$

so $c\mathbf{v}$ is in W.

(b) Let $\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ with $a_4 - a_3 = a_2 - a_1$ be any vector in W. We seek constants c_1, c_2, c_3, c_4 such that

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

which leads to the linear system whose augmented matrix is

$$\left[\begin{array}{rrrr|c} 1 & 0 & 1 & 0 & a_1 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 0 & 1 & 1 & a_3 \\ -1 & 1 & 1 & 1 & a_4 \end{array}\right].$$

When this augmented matrix is transformed to reduced row echelon form we obtain

$$\left[\begin{array}{rrrr|c} 1 & 0 & 0 & -1 & a_1 - a_3 \\ 0 & 1 & 0 & -1 & a_2 - a_3 \\ 0 & 0 & 1 & 1 & a_3 \\ 0 & 0 & 0 & 0 & a_4 + a_1 - a_2 - a_3 \end{array}\right].$$

Since $a_4 + a_1 - a_2 - a_3 = 0$, the system is consistent for any $\mathbf{v}$ in W. Thus $W = \text{span } S$.

(c) A possible answer is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(d) $\begin{bmatrix} 0 \\ 4 \\ 2 \\ 6 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

4. Yes.

5. (a) Let $V = R^2$, $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, $U = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. It follows that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $W \cup U$ and hence $W \cup U$ is not a subspace of V.
   (b) When W is contained in U or U is contained in W.
   (c) Let **u** and **v** be in $W \cap U$ and let c be a scalar. Since vectors **u** and **v** are in both W and U so is $\mathbf{u} + \mathbf{v}$. Thus $\mathbf{u} + \mathbf{v}$ is in $W \cap U$. Similarly, $c\mathbf{u}$ is in W and in U, so it is in $W \cap U$.

6. If $W = R^3$, then it contains the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. If W contains the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then W contains the span of these vectors which is R^3. It follows that $W = R^3$.

7. (a) Yes. (b) They are identical.

8. (a) m arbitrary and $b = 0$. (b) $r = 0$.

9. Suppose that W is a subspace of V. Let **u** and **v** be in W and let r and s be scalars. Then $r\mathbf{u}$ and $s\mathbf{v}$ are in W, so $r\mathbf{u} + s\mathbf{v}$ is in W. Conversely, if $r\mathbf{u} + s\mathbf{v}$ is in W for any **u** and **v** in W and any scalars r and s, then for $r = s = 1$ we have $\mathbf{u} + \mathbf{v}$ is in W. Also, for $s = 0$ we have $r\mathbf{u}$ is in W. Hence, W is a subspace of V.

10. Let **x** and **y** be in W, so that $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$. Then

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \lambda\mathbf{x} + \lambda\mathbf{y} = \lambda(\mathbf{x} + \mathbf{y}).$$

Hence, $\mathbf{x} + \mathbf{y}$ is in W. Also, if r is a scalar, then $A(r\mathbf{x}) = r(A\mathbf{x}) = r(\lambda\mathbf{x}) = \lambda(r\mathbf{x})$, so $r\mathbf{x}$ is in W. Hence W is a subspace of R^n.

12. $a = 1$.

14. (a) One possible answer: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \right\}$.
   (b) One possible answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.
   (c) $\left[\mathbf{v}\right]_S = \begin{bmatrix} 1 \\ \frac{3}{2} \\ -\frac{5}{2} \end{bmatrix}$, $\left[\mathbf{v}\right]_T = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

15. Since S is a linearly independent set, just follow the steps given in the proof of Theorem 3.10.

16. Possible answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

18. (a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. (b) There is no basis.

19. rank A^T = row rank A^T = column rank A = rank A.

20. (a) Theorem 3.16 implies that row space A = row space B. Thus,

$$\text{rank } A = \text{row rank } A = \text{row rank } B = \text{rank } B.$$

(b) This follows immediately since A and B have the same reduced row echelon form.

21. (a) From the definition of a matrix product, the rows of AB are linear combinations of the rows of B. Hence, the row space of AB is a subspace of the row space of B and it follows that rank $(AB) \leq$ rank B. From Exercise 19 above, rank $(AB) \leq$ rank $((AB)^T)$ = rank $(B^T A^T)$. A similar argument shows that rank $(AB) \leq$ rank A^T = rank A. It follows that rank $(AB) \leq \min\{\text{rank } A, \text{rank } B\}$.

(b) One such pair of matrices is $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) Since $A = (AB)B^{-1}$, by (a), rank $A \leq$ rank (AB). But (a) also implies that rank $(AB) \leq$ rank A, so rank (AB) = rank A.

(d) Since $B = A^{-1}(AB)$, by (a), rank $B \leq$ rank (AB). But (a) also implies that rank $(AB) \leq$ rank B, so rank (AB) = rank B.

(e) rank (PAQ) = rank (PA), by part (c), which is rank A, by part (d).

22. (a) Let $q = \dim NS(A)$ and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ be a basis for $NS(A)$. We can extend S to a basis for R^n. Let $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ be a linearly independent subset of R^n such that $\mathbf{v}_1, \ldots, \mathbf{v}_q, \mathbf{w}_1, \ldots, \mathbf{w}_r$ is a basis for R^n. Then $r + q = n$. We need only show that r = rank A. Every vector $\mathbf{v}$ in R^n can be written as

$$\mathbf{v} = \sum_{i=1}^{r} c_i \mathbf{v}_i + \sum_{j=1}^{r} b_j \mathbf{w}_j$$

and since $A\mathbf{v}_i = \mathbf{0}$, $A\mathbf{v} = \displaystyle\sum_{j=1}^{r} b_j A\mathbf{w}_j$. Since $\mathbf{v}$ is an arbitrary vector in R^n, this implies that column space A = span $\{A\mathbf{w}_1, A\mathbf{w}_2, \ldots, A\mathbf{w}_r\}$. These vectors are also linearly independent, because if

$$k_1 A\mathbf{w}_1 + k_2 A\mathbf{w}_2 + \cdots + k_r A\mathbf{w}_r = \mathbf{0}$$

then $\mathbf{w} = \displaystyle\sum_{j=1}^{r} k_j \mathbf{w}_j$ belongs to $NS(A)$. As such it can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$. But since S and span T have only the zero vector in common, $k_j = 0$ for $j = 1, 2, \ldots, r$. Thus, rank $A = r$.

(b) If A is nonsingular then $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{0}$ which implies that $\mathbf{x} = \mathbf{0}$ and thus $\dim NS(A) = 0$. If $\dim NS(A) = 0$ then $NS(A) = \{\mathbf{0}\}$ and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution so A is nonsingular.

23. From Exercise 22, $NS(BA)$ is the set of all vectors $\mathbf{x}$ such that $BA\mathbf{x} = \mathbf{0}$. We first show that if $\mathbf{x}$ is in $NS(BA)$, then $\mathbf{x}$ is in $NS(A)$. If $BA\mathbf{x} = \mathbf{0}$, $B^{-1}(BA\mathbf{x}) = B^{-1}\mathbf{0} = \mathbf{0}$, so $A\mathbf{x} = \mathbf{0}$, which implies that $\mathbf{x}$ is in $NS(A)$. We next show that if $\mathbf{x}$ is in $NS(A)$, then $\mathbf{x}$ is in $NS(BA)$. If $A\mathbf{x} = \mathbf{0}$, then $B(A\mathbf{x}) = B\mathbf{0} = \mathbf{0}$, so $(BA)\mathbf{x} = \mathbf{0}$. Hence, $\mathbf{x}$ is in $NS(BA)$. We conclude that $NS(BA) = NS(A)$.

24. (a) 1. (b) 2.

26. We have $XY^T = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & & \vdots & \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{bmatrix}$.

    Each row of XY^T is a multiple of Y^T, hence rank $XY^T = 1$.

27. Let $\mathbf{x}$ be nonzero. Then $A\mathbf{x} \neq \mathbf{x}$ so $A\mathbf{x} - \mathbf{x} = (A - I_n)\mathbf{x} \neq \mathbf{0}$. That is, there is no nonzero solution to the homogeneous system with square coefficient matrix $A - I_n$. Hence the only solution to the homogeneous system with coefficient matrix $A - I_n$ is the zero solution which implies that $A - I_n$ is nonsingular.

28. Assume rank $A < n$. Then the columns of A are linearly dependent. Hence there exists $\mathbf{x}$ in R^n such that $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$. But then $A^TA\mathbf{x} = \mathbf{0}$ which implies that the homogeneous linear system with coefficient matrix A^TA has a nontrivial solution. This is a contradiction that A^TA is nonsingular, hence the columns of A must be linearly independent. That is, rank $A = n$.

29. (a) Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then rank $A =$ rank $B = 1$ but $A + B = I_2$, so rank $(A + B) = 2$.

    (b) Counterexample: $A = \begin{bmatrix} 1 & -9 \\ 7 & 1 \end{bmatrix}$, $B = -A$. Then rank $A =$ rank $B = 2$ but $A + B = O$, so rank $(A + B) = 0$.

    (c) For A and B as in part (b), rank $(A + B) \neq$ rank $A+$ rank $B = 2 + 2 = 4$.

30. Linearly dependent. Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent in R^n, we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

    where $c_1, c_2, \ldots, c_k$ are not all zero. Then

$$\begin{aligned} A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) &= A\mathbf{0} = \mathbf{0} \\ c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \cdots + c_k(A\mathbf{v}_k) &= \mathbf{0} \end{aligned}$$

    so $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_k$ are linearly dependent.

31. Suppose that the linear system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $m \times 1$ matrix $\mathbf{b}$. Since $A\mathbf{x} = \mathbf{0}$ always has the trivial solution, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Conversely, suppose that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then nullity $A = 0$, so by Theorem 4.19, rank $A = n$. Thus, dim column space $A = n$, so the n columns of A, which span its column space, form a basis for the column space. If $\mathbf{b}$ is an $m \times 1$ matrix then $\mathbf{b}$ is a vector in R^m. If $\mathbf{b}$ is in the column space of A, then $\mathbf{b}$ can be written as a linear combination of the columns of A in one and only one way. That is, $A\mathbf{x} = \mathbf{b}$ has exactly one solution. If $\mathbf{b}$ is not in the column space of A, then $A\mathbf{x} = \mathbf{b}$ has no solution. Thus, $A\mathbf{x} = \mathbf{b}$ has at most one solution.

32. Suppose $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $m \times 1$ matrix $\mathbf{b}$. Then by Exercise 30, the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. That is, nullity $A = 0$. Then rank $A = n-$ nullity $A = n$. So the columns of A are linearly independent. Conversely, if the columns of A are linearly independent, then rank $A = n$, so nullity $A = 0$. This implies that the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Hence, by Exercise 30, $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $m \times 1$ matrix $\mathbf{b}$.

33. Let A be an $m \times n$ matrix whose rank is k. Then the dimension of the solution space of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ is $n - k$, so the general solution to the homogeneous system has $n - k$ arbitrary parameters. As we noted at the end of Section 4.7, every solution $\mathbf{x}$ to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution to the given nonhomogeneous system, and $\mathbf{x}_h$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. Hence, the general solution to the given nonhomogeneous system has $n - k$ arbitrary parameters.

34. Let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{v} = \mathbf{w}'_1 + \mathbf{w}'_2$ be in W, where $\mathbf{w}_1$ and $\mathbf{w}'_1$ are in W_1 and $\mathbf{w}_2$ and $\mathbf{w}'_2$ are in W_2. Then $\mathbf{u} + \mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}'_1 + \mathbf{w}'_2 = (\mathbf{w}_1 + \mathbf{w}'_1) + (\mathbf{w}_2 + \mathbf{w}'_2)$. Since $\mathbf{w}_1 + \mathbf{w}'_1$ is in W_1 and $\mathbf{w}_2 + \mathbf{w}'_2$ is in W_2, we conclude that $\mathbf{u} + \mathbf{v}$ is in W. Also, if c is a scalar, then $c\mathbf{u} = c\mathbf{w}_1 + c\mathbf{w}_2$, and since $c\mathbf{w}_1$ is in W_1, and $c\mathbf{w}_2$ is in W_2, we conclude that $c\mathbf{u}$ is in W.

35. Since $V = W_1 + W_2$, every vector $\mathbf{v}$ in W can be written as $\mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_1$ in W_1 and $\mathbf{w}_2$ in W_2. Suppose now that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{v} = \mathbf{w}'_1 + \mathbf{w}'_2$. Then $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}'_1 + \mathbf{w}'_2$ so

$$\mathbf{w}_1 - \mathbf{w}'_1 = \mathbf{w}'_2 - \mathbf{w}_2 \qquad (*)$$

Since $\mathbf{w}_1 - \mathbf{w}'_1$ is in W_1 and $\mathbf{w}'_2 - \mathbf{w}_2$ is in W_2, $\mathbf{w}_1 - \mathbf{w}'_1$ is in $W_1 \cap W_2 = \{\mathbf{0}\}$. Hence $\mathbf{w}_1 = \mathbf{w}'_1$. Similarly, or from $(*)$ we conclude that $\mathbf{w}_2 = \mathbf{w}'_2$.

36. W must be closed under vector addition and under multiplication of a vector by an arbitrary scalar. Thus, along with $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, W must contain $\displaystyle\sum_{i=1}^{k} a_i\mathbf{v}_i$ for any set of coefficients $a_1, a_2, \ldots, a_k$. Thus W contains span S.

Chapter Review for Chapter 4, p. 288

True or False

1. True.	2. True.	3. False.	4. False.	5. True.	6. False.
7. True.	8. True.	9. True.	10. False.	11. False.	12. True.
13. False.	14. True.	15. True.	16. True.	17. True.	18. True.
19. False.	20. False.	21. True.	22. True.		

Quiz

1. No. Property 1 in Definition 4.4 is not satisfied.

2. No. Properties 5–8 in Definition 4.4 are not satisfied.

3. Yes.

4. No. Property (b) in Theorem 4.3 is not satisfied.

5. If $p(t)$ and $q(t)$ are in W and c is any scalar, then

$$\begin{aligned}(p+q)(0) &= p(0) + q(0) = 0 + 0 = 0\\ (cp)(0) &= cp(0) = c0 = 0.\end{aligned}$$

Hence $p + q$ and cp are in W. Therefore, W is a subspace of P_2. Basis $= \{t^2, t\}$.

6. No. S is linearly dependent.

7. $\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix} \right\}$.

8. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$

9. $\left\{ \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \right\}.$

10. Dimension of null space $= n - \operatorname{rank} A = 3 - 2 = 1$.

11. $\mathbf{x}_p = \begin{bmatrix} -\frac{1}{3} \\ \frac{7}{6} \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_h = r \begin{bmatrix} -1 \\ -\frac{1}{4} \\ \frac{3}{2} \\ 1 \end{bmatrix}$, where r is any number.

12. $c \neq \pm 2$.

Chapter 5

Inner Product Spaces

Section 5.1, p. 297

2. (a) 2. (b) $\sqrt{26}$. (c) $\sqrt{21}$.

4. (a) $3\sqrt{3}$. (b) $3\sqrt{3}$.

6. (a) $\sqrt{155}$. (b) $\sqrt{3}$.

8. $c = \pm 3$.

10. (a) $-\dfrac{32}{\sqrt{14}\sqrt{77}}$. (b) $\dfrac{2}{\sqrt{2}\sqrt{5}}$.

12. $\sqrt{35}$.

13. (a) If $\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, then $\mathbf{u} \cdot \mathbf{u} = a_1^2 + a_2^2 + a_3^2 > 0$ if not all $a_1, a_2, a_3 = 0$.

    $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

    (b) If $\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = \displaystyle\sum_{i=1}^{3} a_i b_i$.

    (c) We have $\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$. Then if $\mathbf{w} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$,

$$\begin{aligned}
(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= (a_1 + b_1)c_1 + (a_2 + b_2)c_2 + (a_3 + b_3)c_3 \\
&= (a_1c_1 + b_1c_1) + (a_2c_2 + b_2c_2) + (a_3c_3 + b_3c_3) \\
&= (a_1c_1 + a_2c_2 + a_3c_3) + (b_1c_1 + b_2c_2 + b_3c_3) \\
&= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}
\end{aligned}$$

    (d) $c\mathbf{u} \cdot \mathbf{v} = (ca_1)b_1 + (ca_2)b_2 + (ca_3)b_3 = c(a_1b_1 + a_2b_2 + a_3b_3) = c(\mathbf{u} \cdot \mathbf{v})$.

14. $\mathbf{u} \cdot \mathbf{u} = 14$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = 15$, $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = 6$, $\mathbf{u} \cdot \mathbf{w} = 0$, $\mathbf{v} \cdot \mathbf{w} = 6$.

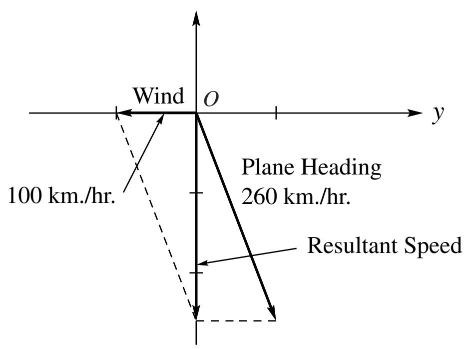
15. (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$; $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$. (b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$.

16. (a) $\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = 1$, etc. (b) $\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} = 0$, etc.

18. (a) $\mathbf{v}_1$ and $\mathbf{v}_2$; $\mathbf{v}_1$ and $\mathbf{v}_3$; $\mathbf{v}_1$ and $\mathbf{v}_4$; $\mathbf{v}_1$ and $\mathbf{v}_6$; $\mathbf{v}_2$ and $\mathbf{v}_3$; $\mathbf{v}_2$ and $\mathbf{v}_5$; $\mathbf{v}_2$ and $\mathbf{v}_6$; $\mathbf{v}_3$ and $\mathbf{v}_5$; $\mathbf{v}_4$ and $\mathbf{v}_5$; $\mathbf{v}_5$ and $\mathbf{v}_6$.
    (b) $\mathbf{v}_1$ and $\mathbf{v}_5$.
    (c) $\mathbf{v}_3$ and $\mathbf{v}_6$.

20. $x = 3 + 0t$, $y = -1 + t$, $z = -3 - 5t$.

22.



Resultant speed: 240 km./hr.

24. $c = 2$.

26. Possible answer: $a = 1$, $b = 0$, $c = -1$.

28. $c = \frac{4}{5}$.

29. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\mathbf{v} = k\mathbf{u}$, so

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot k\mathbf{u}}{\|\mathbf{u}\|\,\|k\mathbf{v}\|} = \frac{k\|\mathbf{u}\|^2}{|k|\,\|\mathbf{u}\|^2} = \pm 1.$$

30. Let $\mathbf{v} = \begin{bmatrix}a\\b\\c\end{bmatrix}$ be a vector in R^3 that is orthogonal to every vector in R^3. Then $\mathbf{v}\cdot\mathbf{i} = 0$ so $\begin{bmatrix}a\\b\\c\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = a = 0$. Similarly, $\mathbf{v}\cdot\mathbf{j} = 0$ and $\mathbf{v}\cdot\mathbf{k} = 0$ imply that $b = c = 0$.

31. Every vector in span $\{\mathbf{w}, \mathbf{x}\}$ is of the form $a\mathbf{w} + b\mathbf{x}$. Then $\mathbf{v}\cdot(a\mathbf{w} + b\mathbf{x}) = a(0) + b(0) = 0$.

32. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be in V, so that $\mathbf{u}\cdot\mathbf{v}_1 = 0$ and $\mathbf{u}\cdot\mathbf{v}_2 = 0$. Let c be a scalar. Then $\mathbf{u}\cdot(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{u}\cdot\mathbf{v}_1 + \mathbf{u}\cdot\mathbf{v}_2 = 0 + 0 = 0$, so $\mathbf{v}_1 + \mathbf{v}_2$ is in V. Also, $\mathbf{u}\cdot(c\mathbf{v}_1) = c(\mathbf{u}\cdot\mathbf{v}_1) = c(0) = 0$, so $c\mathbf{v}_1$ is in V.

33. $\|c\mathbf{x}\| = \sqrt{(cx)^2 + (cy)^2} = \sqrt{c^2}\,\sqrt{x^2 + y^2} = |c|\,\|\mathbf{x}\|$.

34. $\|\mathbf{u}\| \left\| \frac{1}{\|\mathbf{x}\|}\mathbf{x} \right\| = \frac{1}{\|\mathbf{x}\|} \cdot \|\mathbf{x}\| = 1$.

35. Let $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$. Then $(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3)\cdot\mathbf{v}_i = \mathbf{0}\cdot\mathbf{v}_i = 0$ for $i = 1, 2, 3$. Thus, $a_i(\mathbf{v}_i\cdot\mathbf{v}_i) = 0$. Since $\mathbf{v}_i\cdot\mathbf{v}_i \neq 0$ we can conclude that $a_i = 0$ for $i = 1, 2, 3$.

36. We have by Theorem 5.1,

$$\mathbf{u}\cdot(\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{w})\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{u} + \mathbf{w}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}.$$

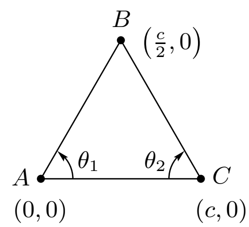
37. (a) $(\mathbf{u}+c\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+(c\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+c(\mathbf{v}\cdot\mathbf{w})$.

    (b) $\mathbf{u}\cdot(c\mathbf{v})=c\mathbf{v}\cdot\mathbf{u}=c(\mathbf{v}\cdot\mathbf{u})=c(\mathbf{u}\cdot\mathbf{v})$.

    (c) $(\mathbf{u}+\mathbf{v})\cdot c\mathbf{w}=\mathbf{u}\cdot(c\mathbf{w})+\mathbf{v}\cdot(c\mathbf{w})=c(\mathbf{u}\cdot\mathbf{w})+c(\mathbf{v}\cdot\mathbf{w})$.

38. Taking the rectangle as suggested, the length of each diagonal is $\sqrt{a^2+b^2}$.

39. Let the vertices of an isosceles triangle be denoted by A, B, C. We show that the cosine of the angles between sides CA and AB and sides AC and CB are the same. (See the figure.)



To simplify the expressions involved let $A(0,0)$, $B(c/2,b)$ and $C(c,0)$. (The perpendicular from B to side AC bisects it. Hence we have the form of a general isosceles triangle.) Let

$$\mathbf{v}=\text{vector from } A \text{ to } B=\begin{bmatrix}\frac{c}{2}\\ b\end{bmatrix}$$

$$\mathbf{w}=\text{vector from } A \text{ to } C=\begin{bmatrix}c\\ 0\end{bmatrix}$$

$$\mathbf{u}=\text{vector from } C \text{ to } B=\begin{bmatrix}-\frac{c}{2}\\ b\end{bmatrix}.$$

Let θ_1 be the angle between $\mathbf{v}$ and $\mathbf{w}$; then

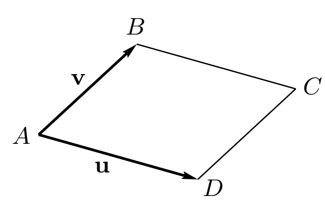
$$\cos\theta_1=\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}=\frac{\frac{c^2}{2}}{\sqrt{\frac{c^2}{4}+b^2}\,\sqrt{c^2}}.$$

Let θ_2 be the angle between $-\mathbf{w}$ and $\mathbf{u}$; then

$$\cos\theta_2=\frac{-\mathbf{w}\cdot\mathbf{u}}{\|\mathbf{w}\|\,\|\mathbf{u}\|}=\frac{\frac{c^2}{2}}{\sqrt{\frac{c^2}{4}+b^2}\,\sqrt{c^2}}.$$

Hence $\cos\theta_1=\cos\theta_2$ implies that $\theta_1=\theta_2$ since an angle θ between vectors lies between 0 and π radians.

40. Let the vertices of a parallelogram be denoted A, B, C, D as shown in the figure. We assign coordinates to the vertices so that the lengths of the opposite sides are equal. Let $(A(0,0)$, $B(t,h)$, $C(s+t,h)$, $D(s,0)$.



Then vectors corresponding to the diagonals are as follows:

The parallelogram is a rhombus provided all sides are equal. Hence we have length $(\overline{AB})$ = length $(\overline{AD})$. It follows that length $(\overline{AD}) = s$ and length $(\overline{AB}) = \sqrt{t^2+h^2}$, thus $s = \sqrt{t^2+h^2}$. To show that the diagonals are orthogonal we show $\mathbf{v}\cdot\mathbf{w} = 0$:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= (s+t)(s-t) - h^2\\ &= s^2 - t^2 - h^2 = s^2 - (t^2+h^2)\\ &= s^2 - s^2 \quad (\text{since } s = \sqrt{t^2+h^2})\\ &= 0.\end{aligned}$$

Conversely, we next show that if the diagonals of a parallelogram are orthogonal then the parallelogram is a rhombus. We show that length $(\overline{AB}) = \sqrt{t^2+h^2} = s =$ length $(\overline{AD})$. Since the diagonals are orthogonal we have $\mathbf{v}\cdot\mathbf{w} = s^2 - (t^2+h^2) = 0$. But then it follows that $s = \sqrt{t^2+h^2}$.

Section 5.2, p. 306

2. (a) $-4\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ (b) $3\mathbf{i}-8\mathbf{j}-\mathbf{k}$ (c) $0\mathbf{i}+0\mathbf{j}+0\mathbf{k}$ (d) $4\mathbf{i}+4\mathbf{j}+8\mathbf{k}$.

4. (a) $\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$
   $\mathbf{v}\times\mathbf{u} = (u_3v_2 - u_2v_3)\mathbf{i} + (v_3u_1 - v_1u_3)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k} = -(\mathbf{u}\times\mathbf{v})$

   (b) $$\begin{aligned}\mathbf{u}\times(\mathbf{v}+\mathbf{w}) &= [u_2(v_3+w_3) - u_3(v_2+w_2)]\mathbf{i}\\ &\quad + [u_3(v_1+w_1) - (u_1(v_3+w_3)]\mathbf{j}\\ &\quad + [u_1(v_2+w_2) - u_2(v_1+w_1)]\mathbf{k}\\ &= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\\ &\quad + (u_2w_3 - u_3w_2)\mathbf{i} + (u_3w_1 - u_1w_3)\mathbf{j} + (u_1w_2 - u_2w_1)\mathbf{k}\\ &= \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}\end{aligned}$$

   (c) Similar to the proof for (b).

   (d) $$\begin{aligned}c(\mathbf{u}\times\mathbf{v}) &= c[(u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}]\\ &= (cu_2v_3 - cu_3v_2)\mathbf{i} + (cu_3v_1 - cu_1v_3)\mathbf{j} + (cu_1v_2 - cu_2v_1)\mathbf{k}\\ &= (c\mathbf{u})\times\mathbf{v}.\end{aligned}$$
   Similarly, $c(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(c\mathbf{v})$.

   (e) $\mathbf{u}\times\mathbf{u} = (u_2u_3 - u_3u_2)\mathbf{i} + (u_3u_1u_3)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} = \mathbf{0}$.

   (f) $\mathbf{0}\times\mathbf{u} = (0u_3 - u_30)\mathbf{i} + (0u_1 - u_10)\mathbf{j} + (0u_2 - u_20)\mathbf{k} = \mathbf{0}$.

   (g) $$\begin{aligned}\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= [u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}]\times[(v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}]\\ &= [u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3)]\mathbf{i}\\ &\quad + [u_3(v_2w_3 - v_3w_2) - u_1(v_1w_2 - v_2w_1)]\mathbf{j}\\ &\quad + [u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2)]\mathbf{k}.\end{aligned}$$
   On the other hand,

   $$(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} = (u_1w_1 + u_2w_2 + u_3w_3)[v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}] - (u_1v_1 + u_2v_2 + u_3v_3)[w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}].$$

   Expanding and simplifying the expression for $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ shows that it is equal to that for $(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$.

   (h) Similar to the proof for (g).

6. (a) $(-15\mathbf{i} - 2\mathbf{j} + 9\mathbf{k})\cdot\mathbf{u} = 0$; $(-15\mathbf{i} - 2\mathbf{j} + 9\mathbf{k})\cdot\mathbf{v} = 0$.

   (b) $(-3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})\cdot\mathbf{u} = 0$; $(-3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})\cdot\mathbf{v} = 0$.

   (c) $(7\mathbf{i} + 5\mathbf{j} - \mathbf{k})\cdot\mathbf{u} = 0$; $(7\mathbf{i} + 5\mathbf{j} - \mathbf{k})\cdot\mathbf{v} = 0$.

   (d) $\mathbf{0}\cdot\mathbf{u} = 0$; $\mathbf{0}\cdot\mathbf{v} = 0$.

7. Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} &= [(u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}]\cdot\mathbf{w}\\ &= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3\\ &\quad \text{(expand and collect terms containing } u_i\text{):}\\ &= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)\\ &= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\end{aligned}$$

8. (a) $\mathbf{u}\cdot\mathbf{v} = 3 = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \sqrt{29}\sqrt{11}\cos\theta \cos\theta = \frac{3}{\sqrt{319}} \quad\Longrightarrow\quad\Longrightarrow\quad \sin\theta = \frac{\sqrt{310}}{\sqrt{319}}$. So

$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{310}$$
$$\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = \sqrt{29}\sqrt{11}\frac{\sqrt{310}}{\sqrt{319}} = \sqrt{310} = \|\mathbf{u}\times\mathbf{v}\|.$$

(b) $\mathbf{u}\cdot\mathbf{v} = 1 = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \sqrt{2}\sqrt{14}\cos\theta \cos\theta = \frac{1}{\sqrt{28}} \quad\Longrightarrow\quad\Longrightarrow\quad \sin\theta = \frac{\sqrt{27}}{\sqrt{28}}$. So

$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{27}$$
$$\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = \sqrt{2}\sqrt{14}\frac{\sqrt{27}}{\sqrt{28}} = \sqrt{27} = \|\mathbf{u}\times\mathbf{v}\|.$$

(c) $\mathbf{u}\cdot\mathbf{v} = 9 = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \sqrt{6}\sqrt{26}\cos\theta \cos\theta = \frac{9}{\sqrt{156}} \quad\Longrightarrow\quad\Longrightarrow\quad \sin\theta = \frac{\sqrt{75}}{\sqrt{156}}$. So

$$\|\mathbf{u}\times\mathbf{v}\| = \sqrt{75}$$
$$\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = \sqrt{6}\sqrt{26}\frac{\sqrt{75}}{\sqrt{156}} = \sqrt{75} = \|\mathbf{u}\times\mathbf{v}\|.$$

(d) $\mathbf{u}\cdot\mathbf{v} = 12 = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \sqrt{6}\sqrt{24}\cos\theta \cos\theta = 1 \quad\Longrightarrow\quad\Longrightarrow\quad \sin\theta = 0$. So

$$\|\mathbf{u}\times\mathbf{v}\| = 0$$
$$\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = \sqrt{6}\sqrt{24}0 = \sqrt{310} = \|\mathbf{u}\times\mathbf{v}\|.$$

9. If $\mathbf{v} = c\mathbf{u}$ for some c, then $\mathbf{u}\times\mathbf{v} = c(\mathbf{u}\times\mathbf{u}) = \mathbf{0}$. Conversely, if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$, the area of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ is 0, and hence that parallelogram is degenerate; $\mathbf{u}$ and $\mathbf{v}$ are parallel.

10. $\|\mathbf{u}\times\mathbf{v}\|^2 + (\mathbf{u}\cdot\mathbf{v})^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(\sin^2\theta + \cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2$.

11. Using property (h) of cross product,

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} + (\mathbf{v}\times\mathbf{w})\times\mathbf{u} + (\mathbf{w}\times\mathbf{u})\times\mathbf{v} =$$
$$[(\mathbf{w}\cdot\mathbf{u})\mathbf{v} - (\mathbf{w}\cdot\mathbf{v})\mathbf{u}] + [(\mathbf{u}\cdot\mathbf{v})\mathbf{w} - (\mathbf{u}\cdot\mathbf{w})\mathbf{v}] + [(\mathbf{v}\cdot\mathbf{w})\mathbf{u} - (\mathbf{v}\cdot\mathbf{u})\mathbf{w}] = \mathbf{0}.$$

12. $\frac{1}{2}\sqrt{478}$.

14. $\sqrt{150}$.

16. 39.

18. (a) $3x - 2y + 4z + 16 = 0$; (b) $y - 3z + 3 = 0$.

20. (a) $x = \frac{8}{13} + 23t$, $y = -\frac{27}{16} + 2t$, $z = 0 + 13t$; (b) $x = 0 + 7t$, $y = -8 + 22t$, $z = 4 + 13t$.

22. $\left(-\frac{17}{5}, \frac{38}{5}, -6\right)$.

24. (a) Not all of a, b and c are zero. Assume that $a \neq 0$. Then write the given equation $ax+by+cz+d=0$ as $a\left[x+\left(\frac{d}{a}\right)\right]+by+cz=0$. This is the equation of the plane passing through the point $\left(-\frac{d}{a},0,0\right)$ and having the vector $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ as normal. If $a=0$ then either $b\neq 0$ or $c\neq 0$. The above argument can be readily modified to handle this case.

(b) Let $\mathbf{u}=(x_1,y_1,z_1)$ and $\mathbf{v}=(x_2,y_2,z_2)$ satisfy the equation of the plane. Then show that $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ satisfy the equation of the plane for any scalar c.

(c) Possible answer: $\left\{\begin{bmatrix}1\\0\\-\frac{1}{2}\end{bmatrix},\begin{bmatrix}0\\1\\\frac{3}{4}\end{bmatrix}\right\}$.

26. $\mathbf{u}\times\mathbf{v}=(u_2v_3-u_3v_2)\mathbf{i}+(u_3v_1-u_1v_3)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}$.
Then

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=(u_2v_3-u_3v_2)w_1+(u_3v_1-u_1v_3)w_2+(u_1v_2-u_2v_1)w_3=\begin{vmatrix}u_1&u_2&u_3\\v_1&v_2&v_3\\w_1&w_2&w_3\end{vmatrix}.$$

28. Computing the determinant we have

$$xy_1+yx_2+x_1y_2-x_2y_1-y_2x-x_1y=0.$$

Collecting terms and factoring we obtain

$$x(y_1-y_2)-y(x_1-x_2)+(x_1y_2-x_2y_1)=0.$$

Solving for y we have

$$\begin{aligned}y&=\frac{y_2-y_1}{x_2-x_1}x-\frac{x_1y_2-x_2y_1}{x_2-x_1}\\&=\frac{y_2-y_1}{x_2-x_1}x-\frac{x_1y_2-y_2x_2+y_2x_2-x_2y_1}{x_2-x_1}\\&=\frac{y_2-y_1}{x_2-x_1}x-\frac{y_2(x_1-x_2)+x_2(y_2-y_1)}{x_2-x_1}\\&=\frac{y_2-y_1}{x_2-x_1}(x-x_2)+y_2\end{aligned}$$

which is the two-point form of the equation of a straight line that goes through points (x_1,y_1) and (x_2,y_2). Now, three points are collinear provided that they are on the same line. Hence a point (x_0,y_0) is collinear with (x_1,y_1) and (x_2,y_2) if it satisfies the equation in (6.1). That is equivalent to saying that (x_0,y_0) is collinear with (x_1,y_1) and (x_2,y_2) provided

$$\begin{vmatrix}x_0&y_0&1\\x_1&y_1&1\\x_2&y_2&1\end{vmatrix}=0.$$

29. Using the row operations $-\mathbf{r}_1+\mathbf{r}_2\to\mathbf{r}_2$, $-\mathbf{r}_1+\mathbf{r}_3\to\mathbf{r}_3$, and $-\mathbf{r}_1+\mathbf{r}_4\to\mathbf{r}_4$ we have

$$0=\begin{vmatrix}x&y&z&1\\x_1&y_1&z_1&1\\x_2&y_2&z_2&1\\x_3&y_3&z_3&1\end{vmatrix}=\begin{vmatrix}x&y&z&1\\x_1-x&y_1-y&z_1-z&1\\x_2-x&y_2-y&z_2-z&1\\x_3-x&y_3-y&z_3-z&1\end{vmatrix}=(-1)\begin{vmatrix}x_1-x&y_1-y&z_1-z\\x_2-x&y_2-y&z_2-z\\x_3-x&y_3-y&z_3-z\end{vmatrix}.$$

Using the row operations $-\mathbf{r}_1 \to \mathbf{r}_1$, $\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2$, and $\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3$, we have

$$\begin{aligned}
0 &= \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} \\
&= (x - x_1)[y_2 - y_1 + z_3 - z_1 - y_3 + y_1 - z_2 + z_1] \\
&\quad + (y - y_1)[z_2 - z_1 + x_3 - x_1 - z_3 + z_1 - x_2 + x_1] \\
&\quad + (z - z_1)[x_2 - x_1 + y_3 - y_1 - x_3 + x_1 - y_2 + y_1] \\
&= (x - x_1)[y_2 - y_3 + z_3 - z_2] \\
&\quad + (y - y_1)[z_2 - z_3 + x_3 - x_2] \\
&\quad + (z - z_1)[x_2 - x_3 + y_3 - y_2]
\end{aligned}$$

This is a linear equation of the form $Ax+By+Cz+D=0$ and hence represents a plane. If we replace (x, y, z) in the original expression by (x_i, y_i, z_i), $i = 1$, 2, or 3, the determinant is zero; hence the plane passes through P_i, $i = 1, 2, 3$.

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1. Similar to proof of Theorem 5.1 (Exercise 13, Section 5.1).

2. (b) $(\mathbf{v}, \mathbf{u}) = a_1b_1 - a_2b_1 - a_1b_2 - 3a_2b_2 = (\mathbf{u}, \mathbf{v})$.

   (c) $\begin{aligned}[t](\mathbf{u} + \mathbf{v}, \mathbf{w}) &= (a_1 + b_1)c_1 - (a_2 + b_2)c_1 - (a_1 + b_1)c_2 + 3(a_2 + b_2)c_2 \\ &= (a_1c_1 - a_2c_1 - a_1c_2 + 3a_2c_2) + (b_1c_1 - b_2c_1 - b_1c_2 + 3b_2c_2) \\ &= (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w}).\end{aligned}$

   (d) $(c\mathbf{u}, \mathbf{v}) = (ca_1)b_1 - (ca_2)b_1 - (ca_1)b_2 + 3(ca_2)b_2 = c(a_1b_1 - a_2b_1 - a_1b_2 + 3a_2b_2) = c(\mathbf{u}, \mathbf{v})$.

3. (a) If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ then $(A, A) = \operatorname{Tr}(A^TA) = \displaystyle\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}^2 \geq 0$. Also $(A, A) = 0$ if and only if $a_{ij} = 0$, that is, if and only if $A = O$.

   (b) If $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ then $(A, B) = \operatorname{Tr}(B^TA)$ and $(B, A) = \operatorname{Tr}(A^TB)$. Now

   $$\operatorname{Tr}(B^TA) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik}^T a_{ki} = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ki} a_{ki},$$

   and

   $$\operatorname{Tr}(A^TB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^T b_{ki} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ki},$$

   so $(A, B) = (B, A)$.

   (c) If $C = \begin{bmatrix} c_{ij} \end{bmatrix}$, then $(A + B, C) = \operatorname{Tr}[C^T(A + B)] = \operatorname{Tr}[C^TA + C^TB] = \operatorname{Tr}(C^TA) + \operatorname{Tr}(C^TB) = (A, C) + (B, C)$.

   (d) $(cA, B) = \operatorname{Tr}(B^T(cA)) = c\operatorname{Tr}(B^TA) = c(A, B)$.

5. Let $\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}$ be vectors in R_2 and let c be a scalar. We define $(\mathbf{u}, \mathbf{v}) = u_1v_1 - u_2v_1 - u_1v_2 + 5u_2v_2$.

   (a) Suppose $\mathbf{u}$ is not the zero vector. Then one of u_1 and u_2 is not zero. Hence

   $$(\mathbf{u}, \mathbf{u}) = u_1u_1 - u_2u_1 - u_1u_2 + 5u_2u_2 = (u_1 - u_2)^2 + 4(u_2)^2 > 0.$$

If $(\mathbf{u},\mathbf{u}) = 0$, then

$$u_1u_1 - u_2u_1 - u_1u_2 + 5u_2u_2 = (u_1 - u_2)^2 + 4(u_2)^2 = 0$$

which implies that $u_1 = u_2 = 0$ hence $\mathbf{u} = \mathbf{0}$. If $\mathbf{u} = \mathbf{0}$, then $u_1 = u_2 = 0$ and

$$(\mathbf{u},\mathbf{u}) = u_1u_1 - u_2u_1 - u_1u_2 + 5u_2u_2 = 0.$$

(b) $(\mathbf{u},\mathbf{v}) = u_1v_1 - u_2v_1 - u_1v_2 + 5u_2v_2 = v_1u_1 - v_2u_1 - v_1u_2 + 5v_2u_2 = (\mathbf{v},\mathbf{u})$

(c)
$$\begin{aligned}(\mathbf{u}+\mathbf{v},\mathbf{w}) &= (u_1+v_1)w_1 - (u_2+v_2)w_2 - (u_1+v_1)w_2 + 5(u_2+v_2)w_2\\ &= u_1w_1 + v_1w_1 - u_2w_2 - v_2w_2 - u_1w_2 - v_1w_2 + 5u_2w_2 + 5v_2w_2\\ &= (u_1w_1 - u_2w_2 - u_1w_2 + 5u_2w_2) + (v_1w_1 - v_2w_2 - v_1w_2 + 5v_2w_2)\\ &= (\mathbf{u},\mathbf{w}) + (\mathbf{v},\mathbf{w})\end{aligned}$$

(d) $(c\mathbf{u},\mathbf{v}) = (cu_1)v_1 - (cu_2)v_1 - (cu_1)v_2 + 5(cu_2)v_2 = c(u_1v_1 - u_2v_1 - u_1v_2 + 5u_2v_2) = c(\mathbf{u},\mathbf{v})$

6. (a) $(p(t),q(t)) = \int_0^1 p(t)^2\,dt \geq 0$. Since $p(t)$ is continuous,

$$\int_0^1 p(t)^2\,dt = 0 \quad\Longleftrightarrow\quad p(t) = 0.$$

(b) $(p(t),q(t)) = \int_0^1 p(t)q(t)\,dt = \int_0^1 q(t)p(t)\,dt = (q(t),p(t))$.

(c) $(p(t)+q(t),r(t)) = \int_0^1 (p(t)+q(t))r(t)\,dt = \int_0^1 p(t)r(t)\,dt + \int_0^1 q(t)r(t)\,dt = (p(t),r(t)) + (q(t),r(t))$.

(d) $(cp(t),q(t)) = \int_0^1 (cp(t))q(t)\,dt = c\int_0^1 p(t)q(t)\,dt = c(p(t),q(t))$.

7. (a) $\mathbf{0}+\mathbf{0} = \mathbf{0}$ so $(\mathbf{0},\mathbf{0}) = (\mathbf{0},\mathbf{0}+\mathbf{0}) = (\mathbf{0},\mathbf{0}) + (\mathbf{0},\mathbf{0})$, and then $(\mathbf{0},\mathbf{0}) = 0$. Hence $\|\mathbf{0}\| = \sqrt{(\mathbf{0},\mathbf{0})} = \sqrt{0} = 0$.

(b) $(\mathbf{u},\mathbf{0}) = (\mathbf{u},\mathbf{0}+\mathbf{0}) = (\mathbf{u},\mathbf{0}) + (\mathbf{u},\mathbf{0})$ so $(\mathbf{u},\mathbf{0}) = 0$.

(c) If $(\mathbf{u},\mathbf{v}) = 0$ for all $\mathbf{v}$ in V, then $(\mathbf{u},\mathbf{u}) = 0$ so $\mathbf{u} = \mathbf{0}$.

(d) If $(\mathbf{u},\mathbf{w}) = (\mathbf{v},\mathbf{w})$ for all $\mathbf{w}$ in V, then $(\mathbf{u}-\mathbf{v},\mathbf{w}) = 0$ and so $\mathbf{u} = \mathbf{v}$.

(e) If $(\mathbf{w},\mathbf{u}) = (\mathbf{w},\mathbf{v})$ for all $\mathbf{w}$ in V, then $(\mathbf{w},\mathbf{u}-\mathbf{v}) = 0$ or $(\mathbf{u}-\mathbf{v},\mathbf{w}) = 0$ for all $\mathbf{w}$ in V. Then $\mathbf{u} = \mathbf{v}$.

8. (a) 7. (b) 0. (c) -9.

10. (a) $\frac{13}{6}$. (b) 3. (c) 4.

12. (a) $\sqrt{22}$. (b) $\sqrt{18}$. (c) 1.

14. (a) $-\frac{1}{2}$. (b) 1. (c) $\frac{4}{7}\sqrt{3}$.

16. $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}) = (\mathbf{u},\mathbf{u}) + 2(\mathbf{u},\mathbf{v}) + (\mathbf{v},\mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u},\mathbf{v}) + (\mathbf{v},\mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u},\mathbf{v}) + \|\mathbf{v}\|^2$, and $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u},\mathbf{u}) - 2(\mathbf{u},\mathbf{v}) + (\mathbf{v},\mathbf{v}) = \|\mathbf{u}\|^2 - 2(\mathbf{u},\mathbf{v}) + (\mathbf{v},\mathbf{v}) = \|\mathbf{u}\|^2 - 2(\mathbf{u},\mathbf{v}) + \|\mathbf{v}\|^2$. Hence $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

17. $\|c\mathbf{u}\| = \sqrt{(c\mathbf{u},c\mathbf{u})} = \sqrt{c^2(\mathbf{u},\mathbf{u})} = \sqrt{c^2}\,\sqrt{(\mathbf{u},\mathbf{u})} = |c|\,\|\mathbf{u}\|$.

18. For Example 3: $[a_1b_1 - a_2b_1 - a_1b_2 + 3a_2b_2]^2 \leq [(a_1-a_2)^2 + 2a_2^2][(b_1-b_2)^2 + b_2^2]$.

For Exercise 3: $[\mathrm{Tr}(B^TA)]^2 \leq \mathrm{Tr}(A^TA)\,\mathrm{Tr}(B^TB)$.

For Example 5: $[a_1^2 - a_2b_1 - a_1b_2 + 5a_2b_2]^2 \leq [a_1^2 - 2a_1a_2 + 5a_2^2][b_1^2 - 2b_1b_2 + 5b_2^2]$.

19. $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}) = (\mathbf{u},\mathbf{u}) + 2(\mathbf{u},\mathbf{v}) + (\mathbf{v},\mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u},\mathbf{v}) + \|\mathbf{v}\|^2$. Thus $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $(\mathbf{u},\mathbf{v}) = 0$.

20. 3.

21. $\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 = \frac{1}{4}(\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}) - \frac{1}{4}(\mathbf{u}-\mathbf{v},\mathbf{u}-\mathbf{v})$
$= \frac{1}{4}[(\mathbf{u},\mathbf{u}) + 2(\mathbf{u},\mathbf{v}) + (\mathbf{v},\mathbf{v})] - \frac{1}{4}[(\mathbf{u},\mathbf{u}) - 2(\mathbf{u},\mathbf{v}) + (\mathbf{v},\mathbf{v})] = (\mathbf{u},\mathbf{v})$.

22. The vectors in (b) are orthogonal.

23. Let W be the set of all vectors in V orthogonal to $\mathbf{u}$. Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in W so that $(\mathbf{u},\mathbf{v}) = 0$ and $(\mathbf{u},\mathbf{w}) = 0$. Then $(\mathbf{u}, r\mathbf{v}+s\mathbf{w}) = r(\mathbf{u},\mathbf{v}) + s(\mathbf{u},\mathbf{w}) = r(0) + s(0) = 0$ for any scalars r and s.

24. Example 3: Let S be the natural basis for R^2; $C = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$.

    Example 5: Let S be the natural basis for R^2; $C = \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}$.

26. (a) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{v}-\mathbf{u}\| \geq 0$.
    (b) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{v}-\mathbf{u}\| = (\mathbf{v}-\mathbf{u},\mathbf{v}-\mathbf{u}) = 0$ if and only if $\mathbf{v}-\mathbf{u} = \mathbf{0}$.
    (c) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{v}-\mathbf{u}\| = \|\mathbf{u}-\mathbf{v}\| = d(\mathbf{v},\mathbf{u})$.
    (d) We have $\mathbf{v}-\mathbf{u} = (\mathbf{w}-\mathbf{u})+(\mathbf{v}-\mathbf{w})$ and $\|\mathbf{v}-\mathbf{u}\| \leq \|\mathbf{w}-\mathbf{u}\|+\|\mathbf{v}-\mathbf{w}\|$ so $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w})+d(\mathbf{w},\mathbf{v})$.

28. (a) $\sqrt{\frac{123}{10}}$. (b) $\sqrt{3}$.

30. Orthogonal: (a). Orthonormal: (c).

32. $3a = -5b$.

34. $a = b = 0$.

36. (a) $5a = -3b$. (b) $b = \dfrac{2a(\cos 1 - 1)}{e(\sin 1 - \cos 1 + 1)}$.

37. We must verify Definition 5.2 for

$$(\mathbf{v},\mathbf{w}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_{ij} b_j = \left[\mathbf{v}\right]_S^T C \left[\mathbf{w}\right]_S .$$

We choose to use the matrix formulation of this inner product which appears in Equation (1) since we can then use matrix algebra to verify the parts of Definition 5.2.

(a) $(\mathbf{v},\mathbf{v}) = \left[\mathbf{v}\right]_S^T C \left[\mathbf{v}\right]_S > 0$ whenever $\left[\mathbf{v}\right]_S \neq \mathbf{0}$ since C is positive definite. $(\mathbf{v},\mathbf{v}) = 0$ if and only if $\left[\mathbf{v}\right]_S = \mathbf{0}$ since A is positive definite. But $\left[\mathbf{v}\right]_S = \mathbf{0}$ is true if and only if $\mathbf{v} = \mathbf{0}$.

(b) $(\mathbf{v},\mathbf{w}) = \left[\mathbf{v}\right]_S^T C \left[\mathbf{w}\right]_S$ is a real number so it is equal to its transpose. That is,

$$\begin{aligned}(\mathbf{v},\mathbf{w}) &= \left[\mathbf{v}\right]_S^T C \left[\mathbf{w}\right]_S = \left(\left[\mathbf{v}\right]_S^T C \left[\mathbf{w}\right]_S\right)^T = \left[\mathbf{w}\right]_S^T C^T \left[\mathbf{v}\right]_S \\ &= \left[\mathbf{w}\right]_S^T C \left[\mathbf{v}\right]_S \quad \text{(since } C \text{ is symmetric)} \\ &= (\mathbf{w},\mathbf{v})\end{aligned}$$

(c) $(\mathbf{u}+\mathbf{v},\mathbf{w}) = \left(\left[\mathbf{u}\right]_S + \left[\mathbf{v}\right]_S\right)^T C \left[\mathbf{w}\right]_S = \left(\left[\mathbf{u}\right]_S^T + \left[\mathbf{v}\right]_S^T\right) C \left[\mathbf{w}\right]_S = \left[\mathbf{u}\right]_S^T C \left[\mathbf{w}\right]_S + \left[\mathbf{v}\right]_S^T C \left[\mathbf{w}\right]_S = (\mathbf{u},\mathbf{w}) + (\mathbf{v},\mathbf{w})$.

(d) $$\begin{aligned}(k\mathbf{v},\mathbf{w}) &= \left[k\mathbf{v}\right]_S^T C \left[\mathbf{w}\right]_S \\ &= k\left[\mathbf{v}\right]_S^T C \left[\mathbf{w}\right]_S \quad \text{(by properties of matrix algebra)} \\ &= k(\mathbf{v},\mathbf{w})\end{aligned}$$

38. From Equation (3) it follows that $(A\mathbf{u}, B\mathbf{v}) = (\mathbf{u}, A^T B\mathbf{v})$.

39. If $\mathbf{u}$ and $\mathbf{v}$ are in R^n, let $\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Then

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} a_i b_i = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

40. (a) If $\mathbf{v}_1$ and $\mathbf{v}_2$ lie in W and c is a real number, then $((\mathbf{v}_1 + \mathbf{v}_2), \mathbf{u}_i) = (\mathbf{v}_1, \mathbf{u}_i) + (\mathbf{v}_2, \mathbf{u}_i) = 0 + 0 = 0$ for $i = 1, 2$. Thus $\mathbf{v}_1 + \mathbf{v}_2$ lies in W. Also $(c\mathbf{v}_1, \mathbf{u}_i) = c(\mathbf{v}_1, \mathbf{u}_i) = c0 = 0$ for $i = 1, 2$. Thus $c\mathbf{v}_1$ lies in W.

(b) Possible answer: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

41. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$. If $\mathbf{u}$ is in span S, then

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k.$$

Let $\mathbf{v}$ be orthogonal to $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$. Then

$$\begin{aligned} (\mathbf{v}, \mathbf{w}) &= (\mathbf{v}, c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k) \\ &= c_1(\mathbf{v}, \mathbf{w}_1) + c_2(\mathbf{v}, \mathbf{w}_2) + \cdots + c_k(\mathbf{v}, \mathbf{w}_k) \\ &= c_1(0) + c_2(0) + \cdots + c_k(0) = 0. \end{aligned}$$

42. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal set, by Theorem 5.4 it is linearly independent. Hence, A is nonsingular. Since S is orthonormal,

$$(\mathbf{v}_i, \mathbf{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

This can be written in terms of matrices as

$$\mathbf{v}_i\mathbf{v}_j^T = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

or as $AA^T = I_n$. Then $A^{-1} = A^T$. Examples of such matrices:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{3} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

43. Since some of the vectors $\mathbf{v}_j$ can be zero, A can be singular.

44. Suppose that A is nonsingular. Let $\mathbf{x}$ be a nonzero vector in R^n. Consider $\mathbf{x}^T(A^TA)\mathbf{x}$. We have $\mathbf{x}^T(A^TA)\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x})$. Let $\mathbf{y} = A\mathbf{x}$. Then we note that $\mathbf{x}^T(A^TA)\mathbf{x} = \mathbf{y}\mathbf{y}^T$ which is positive if $\mathbf{y} \neq \mathbf{0}$. If $\mathbf{y} = \mathbf{0}$, then $A\mathbf{x} = \mathbf{0}$, and since A is nonsingular we must have $\mathbf{x} = \mathbf{0}$, a contradiction. Hence, $\mathbf{y} \neq \mathbf{0}$.

45. Since C is positive definite, for any nonzero vector $\mathbf{x}$ in R^n we have $\mathbf{x}^T C\mathbf{x} > 0$. Multiply both sides of $C\mathbf{x} = k\mathbf{x}$ or the left by $\mathbf{x}^T$ to obtain $\mathbf{x}^T C\mathbf{x} = k\mathbf{x}^T\mathbf{x} > 0$. Since $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^T\mathbf{x} > 0$, so $k > 0$.

46. Let C be positive definite. Using the natural basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ for R^n we find that $\mathbf{e}_i^T C\mathbf{e}_i = a_{ii}$ which must be positive, since C is positive definite.

47. Let C be positive definite. Then if $\mathbf{x}$ is any nonzero vector in R^n, we have $\mathbf{x}^T C\mathbf{x} > 0$. Now let $r = -5$. Then $\mathbf{x}^T(rC)\mathbf{x} < 0$. Hence, rC need not be positive definite.

48. Let B and C be positive definite matrices. Then if $\mathbf{x}$ is any nonzero vector in R^n, we have $\mathbf{x}^T B\mathbf{x} > 0$ and $\mathbf{x}^T C\mathbf{x} > 0$. Now $\mathbf{x}^T(B + C)\mathbf{x} = \mathbf{x}^T B\mathbf{x} + \mathbf{x}^T C\mathbf{x} > 0$, so $B + C$ is positive definite.

49. By Exercise 48, S is closed under addition, but by Exercise 46 it is not closed under multiplication. Hence, S is not a subspace of M_{nn}.

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2. $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-5\\2\\5\end{bmatrix}\right\}$. (b) $\left\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}-\frac{5}{\sqrt{54}}\\\frac{2}{\sqrt{54}}\\\frac{5}{\sqrt{54}}\end{bmatrix}\right\}$.

4. $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$.

6. $\left\{\sqrt{\frac{3}{7}}\,(t+1), \frac{1}{\sqrt{7}}\,(9t-5)\right\}$.

8. $\left\{\sqrt{3}\,t, \dfrac{e^t - 3t}{\sqrt{\frac{e^2}{2} - \frac{7}{2}}}\right\}$.

10. $\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}, \frac{\sqrt{2}}{2}\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$.

12. Possible answer: $\left\{\begin{bmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}\right\}$

14. $\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{3}{\sqrt{12}} & \frac{1}{\sqrt{12}}\end{bmatrix}$.

16. $\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{2}{\sqrt{42}} & \frac{6}{\sqrt{42}}\end{bmatrix}$.

18. $\left\{\frac{1}{\sqrt{42}}\begin{bmatrix}-4\\5\\1\end{bmatrix}\right\}$.

19. Let $\mathbf{v} = \displaystyle\sum_{j=1}^{n} c_j\mathbf{u}_j$. Then

$$(\mathbf{v}, \mathbf{u}_i) = \left(\sum_{j=1}^{n} c_j\mathbf{u}_j, \mathbf{u}_i\right) = \sum_{j=1}^{n} c_j(\mathbf{u}_j, \mathbf{u}_i) = c_i$$

since $(\mathbf{u}_j, \mathbf{u}_i) = 1$ if $j = i$ and 0 otherwise.

20. (a) $\left\{\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{bmatrix}\right\}$.

(b) $\mathbf{u} = \frac{7}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix} + \frac{9}{\sqrt{6}}\begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{bmatrix}$.

21. Let $T = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for an inner product space V. If $\left[\mathbf{v}\right]_T = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$, then $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$. Since $(\mathbf{u}_i, \mathbf{u}_j) = 0$ if $i \neq j$ and 1 if $i = j$, we conclude that

$$\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

22. (a) $\sqrt{14}$. (b) $\left\{\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}-2\\2\\-1\end{bmatrix}\right\}$. (c) $\left[\mathbf{v}\right]_T = \begin{bmatrix}\sqrt{5}\\3\end{bmatrix}$, so $\|\left[\mathbf{v}\right]_T\| = \sqrt{5+9} = \sqrt{14}$.

24. $\left\{1, \sqrt{12}\left(t - \frac{1}{2}\right), \sqrt{180}\left(t^2 - t + \frac{1}{6}\right)\right\}$.

25. (a) Verify that

$$(\mathbf{u}_i, \mathbf{u}_j) = \begin{cases}1, & \text{if } i \neq j\\ 0, & \text{if } i = j.\end{cases}$$

Thus, if $A = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$ and $B = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$, then $(A, B) = \operatorname{Tr}(B^T A) = \operatorname{Tr}\left(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\right) = 0$. If $A = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$, then $(A, A) = \operatorname{Tr}(A^T A) = \operatorname{Tr}\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = 1$.

(b) $\left[\mathbf{v}\right]_S = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$.

26. $\left\{\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}\right\}$.

28. $\begin{bmatrix}2\\-3\\1\end{bmatrix} = \frac{4}{\sqrt{5}}\begin{bmatrix}\frac{1}{\sqrt{5}}\\0\\\frac{2}{\sqrt{5}}\end{bmatrix} - \frac{3}{\sqrt{5}}\begin{bmatrix}-\frac{2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{bmatrix} - 3\begin{bmatrix}0\\1\\0\end{bmatrix}$

30. (a) $Q = \begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{6}}\end{bmatrix} \approx \begin{bmatrix}0.8944 & 0.4082\\-0.4472 & 0.8165\\0 & 0.4082\end{bmatrix}$, $R = \begin{bmatrix}\frac{5}{\sqrt{5}} & -\frac{5}{\sqrt{5}}\\0 & \frac{6}{\sqrt{6}}\end{bmatrix} \approx \begin{bmatrix}2.2361 & -2.2361\\0 & 2.4495\end{bmatrix}$.

(b) $Q = \begin{bmatrix}-\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{bmatrix} \approx \begin{bmatrix}-0.5774 & 0 & 0.8165\\0.5774 & -0.7071 & 0.4082\\0.5774 & 0.7071 & 0.4082\end{bmatrix}$.

$R = \begin{bmatrix}-\sqrt{3} & 0 & 0\\0 & -\sqrt{8} & \sqrt{2}\\0 & 0 & \sqrt{6}\end{bmatrix} \approx \begin{bmatrix}-1.7321 & 0 & 0\\0 & -2.8284 & 1.4142\\0 & 0 & -2.4495\end{bmatrix}$.

(c) $Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} \\ 0 & \frac{1}{\sqrt{6}} & -\frac{5}{\sqrt{30}} \end{bmatrix} \approx \begin{bmatrix} 0.8944 & -0.4082 & -0.1826 \\ 0.4472 & 0.8165 & 0.3651 \\ 0 & 0.4082 & -0.9129 \end{bmatrix}$

$R = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{6} & -\frac{7}{\sqrt{6}} \\ 0 & 0 & -\frac{5}{\sqrt{30}} \end{bmatrix} \approx \begin{bmatrix} 2.2361 & 0 & 0 \\ 0 & 2.4495 & -2.8577 \\ 0 & 0 & -0.9129 \end{bmatrix}.$

31. We have $(\mathbf{u}, c\mathbf{v}) = c(\mathbf{u}, \mathbf{v}) = c(0) = 0$.

32. If $\mathbf{v}$ is in span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ then $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Let $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$. Then $(\mathbf{u}, \mathbf{v}) = a_1(\mathbf{u}, \mathbf{u}_1) + a_2(\mathbf{u}, \mathbf{u}_2) + \cdots + a_n(\mathbf{u}, \mathbf{u}_n) = 0$ since $(\mathbf{u}, \mathbf{u}_i) = 0$ for $i = 1, 2, \ldots, n$.

33. Let W be the subset of vectors in R^n that are orthogonal to $\mathbf{u}$. If $\mathbf{v}$ and $\mathbf{w}$ are in W then $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{w}) = 0$. It follows that $(\mathbf{u}, \mathbf{v} + \mathbf{w}) = (\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{w}) = 0$, and for any scalar c, $(\mathbf{u}, c\mathbf{v}) = c(\mathbf{u}, \mathbf{v}) = 0$, so $\mathbf{v} + \mathbf{w}$ and $c\mathbf{v}$ are in W. Hence, W is a subspace of R^n.

34. Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for Euclidean space V. Form the set $Q = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$. None of the vectors in Q is the zero vector. Since Q contains more than n vectors, Q is a linearly dependent set. Thus one of the vectors is not orthogonal to the preceding ones. (See Theorem 5.4). It cannot be one of the $\mathbf{u}$'s, so at least one of the $\mathbf{v}$'s is not orthogonal to the $\mathbf{u}$'s. Check $\mathbf{v}_1 \cdot \mathbf{u}_j$, $j = 1, \ldots, k$. If all these dot products are zero, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1\}$ is an orthonormal set, otherwise delete $\mathbf{v}_1$. Proceed in a similar fashion with $\mathbf{v}_i$, $i = 2, \ldots, n$ using the largest subset of Q that has been found to be orthogonal so far. What remains will be a set of n orthogonal vectors since Q originally contained a basis for V. In fact, the set will be orthonormal since each of the $\mathbf{u}$'s and $\mathbf{v}$'s originally had length 1.

35. $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for V. Hence $\dim V = k$ and

$$(\mathbf{v}_i, \mathbf{v}_j) = \begin{cases} 0, & \text{if } i \neq j. \\ 1, & \text{if } i = j. \end{cases}$$

Let $T = \{a_1\mathbf{v}_1, a_2\mathbf{v}_2, \ldots, a_k\mathbf{v}_k\}$ where $a_j \neq 0$. To show that T is a basis we need only show that it spans V and then use Theorem 4.12(b). Let $\mathbf{v}$ belong to V. Then there exist scalars c_i, $i = 1, 2, \ldots, k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

Since $a_j \neq 0$, we have

$$\mathbf{v} = \frac{c_1}{a_1}a_1\mathbf{v}_1 + \frac{c_2}{a_2}a_2\mathbf{v}_2 + \cdots + \frac{c_k}{a_k}a_k\mathbf{v}_k$$

so span $T = V$. Next we show that the members of T are orthogonal. Since S is orthogonal we have

$$(a_i\mathbf{v}_i, a_j\mathbf{v}_j) = a_ia_j(\mathbf{v}_i, \mathbf{v}_j) = \begin{cases} 0, & \text{if } i \neq j \\ a_ia_j, & \text{if } i = j. \end{cases}$$

Hence T is an orthogonal set. In order for T to be an orthonormal set we must have $a_ia_j = 1$ for all i and j. This is only possible if all $a_i = 1$.

36. We have

$$\mathbf{u}_i = \mathbf{v}_i + \frac{(\mathbf{u}_i, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}\mathbf{v}_1 + \frac{(\mathbf{u}_i, \mathbf{v}_2)}{(\mathbf{v}_2, \mathbf{v}_2)}\mathbf{v}_2 + \cdots + \frac{(\mathbf{u}_i, \mathbf{v}_{i-1})}{(\mathbf{v}_{i-1}, \mathbf{v}_{i-1})}\mathbf{v}_{i-1}.$$

Then

$$r_{ii} = (\mathbf{u}_i, \mathbf{w}_i) = (\mathbf{v}_i, \mathbf{w}_i) + \frac{(\mathbf{u}_i, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}(\mathbf{v}_1, \mathbf{w}_i) + \frac{(\mathbf{u}_i, \mathbf{v}_2)}{(\mathbf{v}_2, \mathbf{v}_2)}(\mathbf{v}_2, \mathbf{w}_i) + \cdots + \frac{(\mathbf{u}_i, \mathbf{v}_{i-1})}{(\mathbf{v}_{i-1}, \mathbf{v}_{i-1})}(\mathbf{v}_{i-1}, \mathbf{w}_i)$$

because $(\mathbf{v}_i, \mathbf{w}_j) = 0$ for $i \neq j$. Moreover, $\mathbf{w}_i = \frac{1}{\|\mathbf{v}_i\|}\mathbf{v}_i$, so $(\mathbf{v}_i, \mathbf{w}_i) = \frac{1}{\|\mathbf{v}_i\|}(\mathbf{v}_i, \mathbf{v}_i) = \|\mathbf{v}_i\|$.

37. If A is an $n \times n$ nonsingular matrix, then the columns of A are linearly independent, so by Theorem 5.8, A has a QR-factorization.

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2. (a) $\left\{\begin{bmatrix} \frac{7}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}\right\}$ (b) $W^{\perp}$ is the normal to the plane represented by W.

4. $\left\{\begin{bmatrix} \frac{1}{2} \\ -\frac{5}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ \frac{13}{4} \\ 0 \\ 1 \end{bmatrix}\right\}$

6. $\left\{\frac{5}{2}t^4 - \frac{10}{3}t^3 + t^2, 10t^4 - 10t^3 + t, 45t^4 - 40t^3 + 1\right\}$

8. $\left\{\begin{bmatrix} -\frac{4}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} & \frac{4}{3} \\ 0 & 1 \end{bmatrix}\right\}$.

10. Basis for null space of A: $\left\{\begin{bmatrix} -\frac{1}{3} \\ \frac{7}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{7}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}\right\}$

    Basis for row space of A: $\left\{\begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{7}{3} \end{bmatrix}, \begin{bmatrix} 0 & 1 & -\frac{7}{3} & \frac{2}{3} \end{bmatrix}\right\}$.

    Basis for null space of A^T: $\left\{\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}\right\}$.

    Basis for column space of A: $\left\{\begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}\right\}$.

12. (a) $\begin{bmatrix} \frac{7}{5} & \frac{11}{5} & \frac{9}{5} & -\frac{3}{5} \end{bmatrix}$. (b) $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$. (c) $\begin{bmatrix} \frac{1}{10} & \frac{9}{5} & \frac{1}{5} & \frac{31}{10} \end{bmatrix}$.

14. (a) $\begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}$. (b) $\begin{bmatrix} \frac{11}{3} \\ \frac{5}{3} \\ \frac{8}{3} \end{bmatrix}$. (c) $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$.

16. $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

18. $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

20. 2

22. $\dfrac{\pi^2}{3} - 4\cos t + \cos 2t$.

24. The zero vector is orthogonal to every vector in W.

25. If $\mathbf{v}$ is in $V^\perp$, then $(\mathbf{v}, \mathbf{v}) = 0$. By Definition 5.2, $\mathbf{v}$ must be the zero vector. If $W = \{\mathbf{0}\}$, then every vector $\mathbf{v}$ in V is in $W^\perp$ because $(\mathbf{v}, \mathbf{0}) = 0$. Thus $W^\perp = V$.

26. Let $W = \operatorname{span} S$, where $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$. If $\mathbf{u}$ is in $W^\perp$, then $(\mathbf{u}, \mathbf{w}) = 0$ for any $\mathbf{w}$ in W. Hence, $(\mathbf{u}, \mathbf{v}_i) = 0$ for $i = 1, 2, \ldots, m$. Conversely, suppose that $(\mathbf{u}, \mathbf{v}_i) = 0$ for $i = 1, 2, \ldots, m$. Let $\mathbf{w} = \displaystyle\sum_{i=1}^{m} c_i \mathbf{v}_i$ be any vector in W. Then $(\mathbf{u}, \mathbf{w}) = \displaystyle\sum_{i=1}^{m} c_i (\mathbf{u}, \mathbf{v}_i) = 0$. Hence $\mathbf{u}$ is in $W^\perp$.

27. Let $\mathbf{v}$ be a vector in R^n. By Theorem 5.12(a), the column space of A^T is the orthogonal complement of the null space of A. This means that $R^n =$ null space of $A \oplus$ column space of A^T. Hence, there exist unique vectors $\mathbf{w}$ in the null space of A and $\mathbf{u}$ in the column space of A^T so $\mathbf{v} = \mathbf{w} + \mathbf{u}$.

28. Let V be a Euclidean space and W a subspace of V. By Theorem 5.10, we have $V = W \oplus W^\perp$. Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ be a basis for W, so $\dim W = r$, and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s\}$ be a basis for $W^\perp$, so $\dim W^\perp = s$. If $\mathbf{v}$ is in V, then $\mathbf{v} = \mathbf{w} + \mathbf{u}$, where $\mathbf{w}$ is in W and $\mathbf{u}$ is in $W^\perp$. Moreover, $\mathbf{w}$ and $\mathbf{u}$ are unique. Then
$$\mathbf{v} = \sum_{i=1}^{r} a_i \mathbf{w}_i + \sum_{j=1}^{s} b_j \mathbf{u}_j$$
so $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ spans V. We now show that S is linearly independent. Suppose
$$\sum_{i=1}^{r} a_i \mathbf{w}_i + \sum_{j=1}^{s} b_j \mathbf{u}_j = \mathbf{0}.$$
Then $\displaystyle\sum_{i=1}^{r} a_i \mathbf{w}_i = -\sum_{j=1}^{s} b_j \mathbf{u}_j$, so $\displaystyle\sum_{i=1}^{r} a_i \mathbf{w}_i$ lies in $W \cap W^\perp = \{\mathbf{0}\}$. Hence $\displaystyle\sum_{i=1}^{r} a_i \mathbf{w}_i = \mathbf{0}$, and since $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ are linearly independent, $a_1 = a_2 = \cdots = a_r = 0$. Similarly, $b_1 = b_2 = \cdots = b_s = 0$. Thus, S is also linearly independent and is then a basis for V. This means that $\dim V = r + s = \dim W + \dim W^\perp$, and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s$ is a basis for V.

29. If $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is an orthogonal basis for W, then
$$\left\{ \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1, \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2, \ldots, \frac{1}{\|\mathbf{w}_m\|}\mathbf{w}_m \right\}$$
is an orthonormal basis for W, so
$$\begin{aligned} \operatorname{proj}_W \mathbf{v} &= \left(\mathbf{v}, \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1\right) \frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 + \left(\mathbf{v}, \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2\right) \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2 + \cdots + \left(\mathbf{v}, \frac{1}{\|\mathbf{w}_m\|}\mathbf{w}_m\right) \frac{1}{\|\mathbf{w}_m\|}\mathbf{w}_m \\ &= \frac{(\mathbf{v}, \mathbf{w}_1)}{(\mathbf{w}_1, \mathbf{w}_1)}\mathbf{w}_1 + \frac{(\mathbf{v}, \mathbf{w}_2)}{(\mathbf{w}_2, \mathbf{w}_2)}\mathbf{w}_2 + \cdots + \frac{(\mathbf{v}, \mathbf{w}_m)}{(\mathbf{w}_m, \mathbf{w}_m)}\mathbf{w}_m. \end{aligned}$$

Section 5.6, p. 356

1. From Equation (1), the normal system of equations is $A^T A \widehat{\mathbf{x}} = A^T \mathbf{b}$. Since A is nonsingular so is A^T and hence so is $A^T A$. It follows from matrix algebra that $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$ and multiplying both sides of the preceding equation by $(A^T A)^{-1}$ gives
$$\widehat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b} = A^{-1}(A^T)^{-1} A^T \mathbf{b} = A^{-1}\mathbf{b}.$$

2. $\widehat{\mathbf{x}} = \begin{bmatrix} \frac{24}{17} \\ -\frac{8}{17} \end{bmatrix} \approx \begin{bmatrix} 1.4118 \\ -0.4706 \end{bmatrix}$.

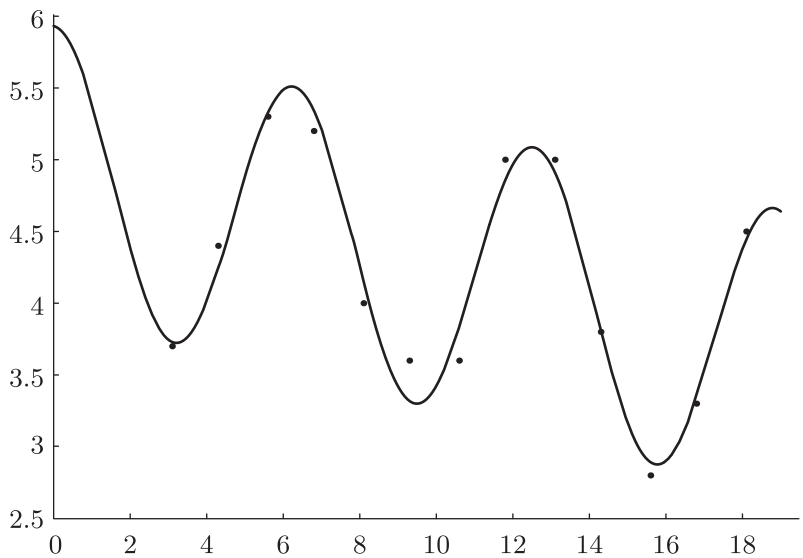
4. Using MATLAB, we obtain

$$Q = \begin{bmatrix} -0.8165 & 0.3961 & 0.4022 & -0.1213 \\ -0.4082 & -0.0990 & -0.5037 & 0.7549 \\ 0 & -0.5941 & 0.7029 & 0.3911 \\ 0.4082 & 0.6931 & 0.3007 & 0.5124 \end{bmatrix}, \quad R = \begin{bmatrix} -2.4495 & -0.4082 \\ 0 & 1.6833 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1.4118 \\ -0.4706 \end{bmatrix}.$$

6. $\mathbf{y} = 1.87 + 1.345t$, $\|\mathbf{e}\| = 1.712$.

7. Minimizing E_2 amounts to searching over the vector space P_2 of all quadratics in order to determine the one whose coefficients give the smallest value in the expression E_2. Since P_1 is a subspace of P_2, the minimization of E_2 has already searched over P_1 and thus the minimum of E_1 cannot be smaller than the minimum of E_2.

8. $y(t) = 4.9345 - 0.0674t + 0.9970\cos t$.



9. $x_1 \approx 4.9345$, $x_2 \approx -6.7426 \times 10^{-2}$, $x_3 \approx 9.9700 \times 10^{-1}$.

10. Let x be the number of years since 1960 ($x = 0$ is 1960).
   (a) $y = 127.871022x - 251292.9948$
   (b) In 2008, expenditure prediction = 5484 in whole dollars.
   In 2010, expenditure prediction = 5740 in whole dollars.
   In 2015, expenditure prediction = 6379 in whole dollars.

12. Let x be the number of years since 1996 ($x = 0$ is 1996).
   (a) $y = 147.186x^2 - 572.67x + 20698.4$
   (b) Compare with the linear regression: $y = 752x + 18932.2$. $E_1 \approx 1.4199 \times 10^7$, $E_2 \approx 2.7606 \times 10^6$.

Supplementary Exercises for Chapter 5, p. 358

1. $(\mathbf{u}, \mathbf{v}) = x_1 - 2x_2 + x_3 = 0$; choose $x_2 = s$, $x_3 = t$. Then $x_1 = 2s - t$ and any vector of the form

$$\begin{bmatrix} 2s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

is orthogonal to $\mathbf{u}$. Hence,

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the subspace of vectors orthogonal to $\mathbf{u}$.

2. Possible answer: $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$

4. Possible answer: $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$

6. $\dfrac{1}{\sqrt{14}}$

7. If $n \neq m$, then

$$\int_0^\pi \sin(mt)\,\sin(nt)\,dt = \left[\frac{\sin(m-n)t}{2(m-n)} - \frac{\sin(m+n)t}{2(m+n)} \right]_0^\pi = 0.$$

This follows since $m-n$ and $m+n$ are integers and sine is zero at integer multiples of π.

8. (a) $-4\cos t + 2\sin t$. (b) $\dfrac{8}{3\pi}\sin t$. (c) $\dfrac{1+\cos 2t}{2}$.

10. (a) $Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{14}} & \frac{4}{\sqrt{21}} \\ -\frac{1}{\sqrt{6}} & \frac{3}{\sqrt{14}} & \frac{2}{\sqrt{21}} \\ \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{21}} \end{bmatrix} \approx \begin{bmatrix} 0.4082 & -0.2673 & 0.8729 \\ -0.4082 & 0.8018 & 0.4364 \\ 0.8165 & 0.5345 & -0.2182 \end{bmatrix}$

$R = \begin{bmatrix} \frac{6}{\sqrt{6}} & \frac{3}{\sqrt{6}} & -\frac{3}{\sqrt{6}} \\ 0 & \frac{7}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ 0 & 0 & \frac{9}{\sqrt{21}} \end{bmatrix} \approx \begin{bmatrix} 2.4495 & 1.2247 & -1.2247 \\ 0 & 1.8708 & 0.8018 \\ 0 & 0 & 1.9640 \end{bmatrix}.$

(b) $Q = \begin{bmatrix} \frac{2}{3} & \frac{5}{\sqrt{90}} \\ -\frac{1}{3} & -\frac{4}{\sqrt{90}} \\ -\frac{2}{3} & \frac{7}{\sqrt{90}} \end{bmatrix} \approx \begin{bmatrix} 0.6667 & 0.5270 \\ -0.3333 & -0.4216 \\ -0.6667 & 0.7379 \end{bmatrix}$

$R = \begin{bmatrix} 3 & -1 \\ 0 & \frac{10}{\sqrt{10}} \end{bmatrix} \approx \begin{bmatrix} 3.0000 & -1.0000 \\ 0 & 3.1623 \end{bmatrix}.$

12. (a) The subspace of R^3 with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{10}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{2}{7} \end{bmatrix} \right\}.$

(b) The subspace of R_4 with basis $\left\{ \begin{bmatrix} 1 & 0 & \frac{5}{3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -\frac{4}{3} & -2 \end{bmatrix} \right\}.$

14. (a) $\frac{3}{5}t$. (b) $\frac{3}{\pi}t$. (c) $-\frac{15}{2\pi^2}(3t^2-1)$.

16. $\sqrt{2\pi}$.

17. Let $\mathbf{u} = \mathrm{col}_i(I_n)$. Then $1 = (\mathbf{u},\mathbf{u}) = (\mathbf{u}, A\mathbf{u}) = a_{ii}$, and thus, the diagonal entries of A are equal to 1. Now let $\mathbf{u} = \mathrm{col}_i(I_n) + \mathrm{col}_j(I_n)$ with $i \neq j$. Then

$$(\mathbf{u},\mathbf{u}) = (\mathrm{col}_i(I_n), \mathrm{col}_i(I_n)) + (\mathrm{col}_j(I_n), \mathrm{col}_j(I_n)) = 2$$

and

$$(\mathbf{u}, A\mathbf{u}) = (\mathrm{col}_i(I_n) + \mathrm{col}_j(I_n), \mathrm{col}_i(A) + \mathrm{col}_j(A)) = a_{ii} + a_{jj} + a_{ij} + a_{ji} = 2 + 2a_{ij}$$

since A is symmetric. It then follows that $a_{ij} = 0$, $i \neq j$. Thus, $A = I_n$.

18. (a) This follows directly from the definition of positive definite matrices.
    (b) This follows from the discussion in Section 5.3 following Equation (5) where it is shown that every positive definite matrix is nonsingular.
    (c) Let $\mathbf{e}_i$ be the ith column of I_n. Then if A is diagonal we have $\mathbf{e}_i^T A\mathbf{e}_i = a_{ii}$. It follows immediately that A is positive semidefinite if and only if $a_{ii} \geq 0$, $i = 1, 2, \ldots, n$.

19. (a) $\|P\mathbf{x}\| = \sqrt{(P\mathbf{x}, P\mathbf{x})} = \sqrt{(P\mathbf{x})^T P\mathbf{x}} = \sqrt{\mathbf{x}^T P^T P\mathbf{x}} = \sqrt{\mathbf{x}^T I_n\mathbf{x}} = \sqrt{\mathbf{x}^T\mathbf{x}} = \|\mathbf{x}\|$.
    (b) Let θ be the angle between $P\mathbf{x}$ and $P\mathbf{y}$. Then, using part (a), we have

$$\cos\theta = \frac{(P\mathbf{x}, P\mathbf{y})}{\|P\mathbf{x}\|\,\|P\mathbf{y}\|} = \frac{(P\mathbf{x})^T P\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{\mathbf{x}^T P^T P\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}.$$

    But this last expression is the cosine of the angle between $\mathbf{x}$ and $\mathbf{y}$. Since the angle is restricted to be between 0 and π we have that the two angles are equal.

20. If A is skew symmetric then $A^T = -A$. Note that $\mathbf{x}^T A\mathbf{x}$ is a scalar, thus $(\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A\mathbf{x}$. That is, $\mathbf{x}^T A\mathbf{x} = (\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A^T\mathbf{x} = -(\mathbf{x}^T A\mathbf{x})$. The only scalar equal to its negative is zero. Hence $\mathbf{x}^T A\mathbf{x} = 0$ for all $\mathbf{x}$.

21. (a) The columns $\mathbf{b}_j$ are in R^m. Since the columns are orthonormal they are linearly independent. There can be at most m linearly independent vectors in R^m. Thus $m \geq n$.
    (b) We have

$$\mathbf{b}_i^T\mathbf{b}_j = \begin{cases} 0, & \text{for } i \neq j \\ 1, & \text{for } i = j. \end{cases}$$

    It follows that $B^T B = I_n$, since the (i, j) element of $B^T B$ is computed by taking row i of B^T times column j of B. But row i of B^T is just $\mathbf{b}_i^T$ and column j of B is $\mathbf{b}_j$.

22. Let $\mathbf{x}$ be in S. Then we can write $\mathbf{x} = \displaystyle\sum_{j=1}^{k} c_j\mathbf{u}_j$. Similarly if $\mathbf{y}$ is in T, we have $\mathbf{y} = \displaystyle\sum_{i=k+1}^{n} c_i\mathbf{u}_i$. Then

$$(\mathbf{x},\mathbf{y}) = \left(\sum_{j=1}^{k} c_j\mathbf{u}_j, \mathbf{y}\right) = \sum_{j=1}^{k} c_j(\mathbf{u}_j, \mathbf{y}) = \sum_{j=1}^{k} c_j\left(\mathbf{u}_j, \sum_{i=k+1}^{n} c_i\mathbf{u}_i\right) = \sum_{j=1}^{k} c_j\left(\sum_{i=k+1}^{n} c_i(\mathbf{u}_j, \mathbf{u}_i)\right).$$

Since $j \neq i$, $(\mathbf{u}_j, \mathbf{u}_i) = 0$, hence $(\mathbf{x},\mathbf{y}) = 0$.

23. Let $\dim V = n$ and $\dim W = r$. Since $V = W \oplus W^\perp$ by Exercise 28, Section 5.5 $\dim W^\perp = n - r$. First, observe that if $\mathbf{w}$ is in W, then $\mathbf{w}$ is orthogonal to every vector in $W^\perp$, so $\mathbf{w}$ is in $(W^\perp)^\perp$. Thus, W is a subspace of $(W^\perp)^\perp$. Now again by Exercise 28, $\dim(W^\perp)^\perp = n - (n - r) = r = \dim W$. Hence $(W^\perp)^\perp = W$.

24. If $\mathbf{u}$ is orthogonal to every vector in S, then $\mathbf{u}$ is orthogonal to every vector in V, so $\mathbf{u}$ is in $V^\perp = \{\mathbf{0}\}$. Hence, $\mathbf{u} = \mathbf{0}$.

25. We must show that the rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ of AA^T are linearly independent. Consider

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \cdots + a_m\mathbf{v}_m = \mathbf{0}$$

which can be written in matrix form as $\mathbf{x}A = \mathbf{0}$ where $\mathbf{x} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix}$. Multiplying this equation on the right by A^T we have $\mathbf{x}AA^T = \mathbf{0}$. Since AA^T is nonsingular, Theorem 2.9 implies that $\mathbf{x} = \mathbf{0}$, so $a_1 = a_2 = \cdots = a_m = 0$. Hence rank $A = m$.

26. We have

$$0 = ((\mathbf{u} - \mathbf{v}), (\mathbf{u} + \mathbf{v})) = (\mathbf{u}, \mathbf{u}) + (\mathbf{u}, \mathbf{v}) - (\mathbf{v}, \mathbf{u}) - (\mathbf{v}, \mathbf{v}) = (\mathbf{u}, \mathbf{u}) - (\mathbf{v}, \mathbf{v}).$$

Therefore $(\mathbf{u}, \mathbf{u}) = (\mathbf{v}, \mathbf{v})$ and hence $\|\mathbf{u}\| = \|\mathbf{v}\|$.

27. Let $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ and $\mathbf{w} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$. By Exercise 26 in Section 4.3, $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$. Then

$$\begin{aligned} d(\mathbf{v}, \mathbf{w}) &= \|\mathbf{v} - \mathbf{w}\| = \sqrt{(\mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w})} \\ &= \sqrt{((a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n, (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n)} \\ &= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2} \end{aligned}$$

since $(\mathbf{v}_i, \mathbf{v}_j) = 0$ if $i \neq j$ and 1 if $i = j$.

28. (a) $\left\|\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\|_1 = 5$; $\left\|\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\|_2 = \sqrt{13}$; $\left\|\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\|_\infty = 3$.

(b) $\left\|\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right\|_1 = 2$; $\left\|\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right\|_2 = 2$; $\left\|\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right\|_\infty = 2$.

(c) $\left\|\begin{bmatrix} -4 \\ -1 \end{bmatrix}\right\|_1 = 5$; $\left\|\begin{bmatrix} -4 \\ -1 \end{bmatrix}\right\|_2 = \sqrt{17}$; $\left\|\begin{bmatrix} -4 \\ -1 \end{bmatrix}\right\|_\infty = 4$.

30. $\|\mathbf{x}\|_1 = |x_1 + |x_2| + \cdots + |x_n| \geq 0$; $\|\mathbf{x}\| = 0$ if and only if $|x_i| = 0$ for $i = 1, 2, \ldots, n$ if and only if $\mathbf{x} = \mathbf{0}$.

$$\|c\mathbf{x}\|_1 = |cx_1| + |cx_2| + \cdots + |cx_n| = |c|\,|x_1| + |c|\,|x_2| + \cdots + |c|\,|x_n| = |c|(|x_1| + |x_2| + \cdots + |x_n|) = |c|\,\|\mathbf{x}\|_1.$$

Let $\mathbf{x}$ and $\mathbf{y}$ be in R^n. By the Triangle Inequality, $|x_i + y_i| \leq |x_i| + |y_i|$ for $i = 1, 2, \ldots, n$. Therefore

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\| &= |x_1 + y_1| + \cdots + |x_n + y_n| \\ &\leq |x_1| + |y_1| + \cdots + |x_n| + |y_n| \\ &= (|x_1| + \cdots + |x_n|) + (|y_1| + \cdots + |y_n|) \\ &= \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1. \end{aligned}$$

Thus $\|\;\|$ is a norm.

31. (a) $\|\mathbf{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\} \geq 0$ since each of $|x_1|, \ldots, |x_n|$ is ≥ 0. Clearly, $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(b) If c is any real scalar

$$\|c\mathbf{x}\|_\infty = \max\{|cx_1|, \ldots, |cx_n|\} = \max\{|c|\,|x_1|, \ldots, |c|\,|x_n|\} = |c| \max\{|x_1|, \ldots, |x_n|\} = |c|\,\|\mathbf{x}\|_\infty.$$

(c) Let $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$ and let

$$\begin{aligned} \|\mathbf{x}\|_\infty &= \max\{|x_1|, \ldots, |x_n|\} = |x_s| \\ \|\mathbf{y}\|_\infty &= \max\{|y_1|, \ldots, |y_n|\} = |y_t| \end{aligned}$$

for some s, t, where $1 \le s \le n$ and $1 \le t \le n$. Then for $i = 1, \dots, n$, we have using the triangle inequality:

$$|x_i + y_i| \le |x_i| + |y_i| \le |x_s| + |y_t|.$$

Thus

$$\|\mathbf{x} + \mathbf{y}\| = \max\{|x_1 + y_1|, \dots, |x_n + y_n|\} \le |x_s| + |y_t| = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.$$

32. (a) Let $\mathbf{x}$ be in R^n. Then

$$\begin{aligned}\|\mathbf{x}\|_2^2 = x_1^2 + \cdots + x_n^2 &\le x_1^2 + \cdots + x_n^2 + 2|x_1|\,|x_2| + \cdots + 2|x_{n-1}|\,|x_n| \\ &= (|x_1| + \cdots + |x_n|)^2 \\ &= \|\mathbf{x}\|_1^2.\end{aligned}$$

(b) Let $|x_i| = \max\{|x_1|, \dots, |x_n|\}$. Then

$$\|\mathbf{x}\|_\infty = |x_i| \le |x_1| + \cdots + |x_n| = \|\mathbf{x}\|_1.$$

Now $\|\mathbf{x}\|_1 = |x_1| + \cdots + |x_n| \le |x_i| + \cdots + |x_i| = n|x_i|$. Hence

$$\frac{\|\mathbf{x}\|_1}{n} \le |x_i| = \|\mathbf{x}\|_\infty.$$

Therefore

$$\frac{\|\mathbf{x}\|_1}{n} \le \|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_1.$$

Chapter Review for Chapter 5, p. 360

True or False

1. True.	2. False.	3. False.	4. False.	5. True.	6. True.
7. False.	8. False.	9. False.	10. False.	11. True.	12. True.

Quiz

1. $b = \frac{\sqrt{2}}{2}$, $c = \pm\frac{\sqrt{2}}{2}$.

2. $\mathbf{x} = \begin{bmatrix} r - 4s \\ -3r + 6s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 6 \\ 0 \\ 1 \end{bmatrix}$, where r and s are any numbers.

3. $p(t) = a + bt$, where $a = -\frac{5}{9}b$ and b is any number.

4. (a) The inner product of $\mathbf{u}$ and $\mathbf{v}$ is bounded by the product of the lengths of $\mathbf{u}$ and $\mathbf{v}$.

   (b) The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ lies between -1 and 1.

5. (a) $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

   (b) Normalize the vectors in S: $\frac{1}{2}\mathbf{v}_1$, $\frac{1}{\sqrt{6}}\mathbf{v}_2$, $\frac{1}{\sqrt{12}}\mathbf{v}_3$.

   (c) Possible answer: $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

6. (b) $\mathbf{w} = \frac{5}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3$.

(c) $\operatorname{proj}_W \mathbf{w} = \begin{bmatrix} -\frac{1}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}$.

Distance from V to $\mathbf{w} = \dfrac{2\sqrt{6}}{3}$.

7. $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

8. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

9. Form the matrix A whose columns are the vectors in S. Find the row reduced echelon form of A. The columns of this matrix can be used to obtain a basis for W. The rows of this matrix give the solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ and from this we can find a basis for $W^\perp$.

10. We have

$$\begin{aligned} \operatorname{proj}_W(\mathbf{u} + \mathbf{v}) &= (\mathbf{u} + \mathbf{v}, \mathbf{w}_1) + (\mathbf{u} + \mathbf{v}, \mathbf{w}_2) + (\mathbf{u} + \mathbf{v}, \mathbf{w}_3) \\ &= (\mathbf{u}, \mathbf{w}_1) + (\mathbf{v}, \mathbf{w}_1) + (\mathbf{u}, \mathbf{w}_2) + (\mathbf{v}, \mathbf{w}_2) + (\mathbf{u}, \mathbf{w}_3) + (\mathbf{v}, \mathbf{w}_3) \\ &= (\mathbf{u}, \mathbf{w}_1) + (\mathbf{u}, \mathbf{w}_2) + (\mathbf{u}, \mathbf{w}_3) + (\mathbf{v}, \mathbf{w}_1) + (\mathbf{v}, \mathbf{w}_2) + (\mathbf{v}, \mathbf{w}_3) \\ &= \operatorname{proj}_W \mathbf{u} + \operatorname{proj}_W \mathbf{v}. \end{aligned}$$

Chapter 6

Linear Transformations and Matrices

Section 6.1, p. 372

2. Only (c) is a linear transformation.

4. (a)

6. If L is a linear transformation then $L(a\mathbf{u} + b\mathbf{v}) = L(a\mathbf{u}) + L(b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v})$. Conversely, if the condition holds let $a = b = 1$; then $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$, and if we let $b = 0$ then $L(a\mathbf{u}) = aL(\mathbf{u})$.

8. (a) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$.

10. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

12. (a) $\begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$. (b) $\begin{bmatrix} -x_2 + 2x_3 \\ -2x_1 + x_2 + 3x_3 \\ x_1 + 2x_2 - 3x_3 \end{bmatrix}$.

14. (a) $\begin{bmatrix} 8 & 5 \end{bmatrix}$. (b) $\begin{bmatrix} \dfrac{-a_1 + 3a_2}{2} & \dfrac{-5a_1 + a_2}{2} \end{bmatrix}$.

16. We have

$$\begin{aligned} L(X + Y) &= A(X + Y) - (X + Y)A = AX + AY - XA - YA \\ &= (AX - XA) + (AY - YA) \\ &= L(X) + L(Y). \end{aligned}$$

Also, $L(aX) = A(aX) - (aX)A = a(AX - XA) = aL(X)$.

18. We have

$$L(\mathbf{v}_1 + \mathbf{v}_2) = (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = (\mathbf{v}_1, \mathbf{w}) + (\mathbf{v}_2, \mathbf{w}) = L(\mathbf{v}_1) + L(\mathbf{v}_2).$$

Also, $L(c\mathbf{v}) = (c\mathbf{v}, \mathbf{w}) = c(\mathbf{v}, \mathbf{w}) = cL(\mathbf{v})$.

20. (a) $17t - 7$. (b) $\left(\dfrac{5a - b}{2}\right)t + \dfrac{a + 5b}{2}$.

21. We have

$$L(\mathbf{u}+\mathbf{v}) = \mathbf{0}_W = \mathbf{0}_W + \mathbf{0}_W = L(\mathbf{u}) + L(\mathbf{v})$$

and

$$L(c\mathbf{u}) = \mathbf{0}_W = c\mathbf{0}_W = cL(\mathbf{u}).$$

22. We have

$$L(\mathbf{u}+\mathbf{v}) = \mathbf{u}+\mathbf{v} = L(\mathbf{u}) + L(\mathbf{v})$$

and

$$L(c\mathbf{u}) = c\mathbf{u} = cL(\mathbf{u}).$$

23. Yes:

$$\begin{aligned} L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= L\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\ &= (a_1+a_1)+(d_1+d_2) \\ &= (a_1+d_1)+(a_2+d_2) \\ &= L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right). \end{aligned}$$

Also, if k is any real number

$$L\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = L\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = ka+kd = k(a+d) = kL\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

24. We have

$$L(f+g) = (f+g)' = f'+g' = L(f)+L(g)$$

and

$$L(af) = (af)' = af' = aL(f).$$

25. We have

$$L(f+g) = \int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = L(f)+L(g)$$

and

$$L(cf) = \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx = cL(f).$$

26. Let X, Y be in M_{nn} and let c be any scalar. Then

$$\begin{aligned} L(X+Y) &= A(X+Y) = AX+AY = L(X)+L(Y) \\ L(cX) &= A(cX) = c(AX) = cL(X) \end{aligned}$$

Therefore, L is a linear transformation.

27. No.

28. No.

29. We have by the properties of coordinate vectors discussed in Section 4.8, $L(\mathbf{u}+\mathbf{v}) = \left[\mathbf{u}+\mathbf{v}\right]_S = \left[\mathbf{u}\right]_S + \left[\mathbf{v}\right]_S = L(\mathbf{u})+L(\mathbf{v})$ and $L(c\mathbf{u}) = \left[c\mathbf{u}\right]_S = c\left[\mathbf{u}\right]_S = cL(\mathbf{u})$.

30. Let $\mathbf{v} = \begin{bmatrix} a & b & c & d \end{bmatrix}$ and we write $\mathbf{v}$ as a linear combination of the vectors in S:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = \mathbf{v} = \begin{bmatrix} a & b & c & d \end{bmatrix}.$$

The resulting linear system has the solution

$$\begin{aligned} a_1 &= 4a + 5b - 3c - 4d \qquad & a_2 &= 2a + 3b - 2c - 2d \\ a_3 &= -a - b + c + d & a_4 &= -3a - 5b + 3c + 4d. \end{aligned}$$

Then $L\left(\begin{bmatrix} a & b & c & d \end{bmatrix}\right) = \begin{bmatrix} -2a - 5b + 3c + 4d & 14a + 19b - 12c - 14d \end{bmatrix}$.

31. Let $L(\mathbf{v}_i) = \mathbf{w}_i$. Then for any $\mathbf{v}$ in V, express $\mathbf{v}$ in terms of the basis vectors of S;

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

and define $L(\mathbf{v}) = \displaystyle\sum_{i=1}^{n} a_i\mathbf{w}_i$. If $\mathbf{v} = \displaystyle\sum_{i=1}^{n} a_i\mathbf{v}_i$ and $\mathbf{w} = \displaystyle\sum_{i=1}^{n} b_i\mathbf{v}_i$ are any vectors in V and c is any scalar, then

$$L(\mathbf{v} + \mathbf{w}) = L\left(\sum_{i=1}^{n}(a_i + b_i)\right)\mathbf{v}_i = \sum_{i=1}^{n}(a_i + b_i)\mathbf{w}_i = \sum_{i=1}^{n} a_i\mathbf{w}_i + \sum_{i=1}^{n} b_i\mathbf{w}_i = L(\mathbf{v}) + L(\mathbf{w})$$

and in a similar fashion

$$L(c\mathbf{v}) = \sum_{i=1}^{n} ca_i\mathbf{w}_i = c\sum_{i=1} a_i\mathbf{w}_i = cL(\mathbf{v})$$

for any scalar c, so L is a linear transformation.

32. Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be in $L(V_1)$ and let c be a scalar. Then $\mathbf{w}_1 = L(\mathbf{v}_1)$ and $\mathbf{w}_2 = L(\mathbf{v}_2)$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are in V_1. Then $\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1) + L(\mathbf{v}_2) = L(\mathbf{v}_1 + \mathbf{v}_2)$ and $c\mathbf{w}_1 = cL(\mathbf{v}_1) = L(c\mathbf{v}_1)$. Since $\mathbf{v}_1 + \mathbf{v}_2$ and $c\mathbf{v}_1$ are in V_1, we conclude that $\mathbf{w}_1 + \mathbf{w}_2$ and $c\mathbf{w}_1$ lie in $L(V_1)$. Hence $L(V_1)$ is a subspace of V.

33. Let $\mathbf{v}$ be any vector in V. Then

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

We now have

$$\begin{aligned} L_1(\mathbf{v}) &= L_1(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) \\ &= c_1L_1(\mathbf{v}_1) + c_2L_1(\mathbf{v}_2) + \cdots + c_nL_1(\mathbf{v}_n) \\ &= c_1L_2(\mathbf{v}_1) + c_2L_2(\mathbf{v}_2) + \cdots + c_nL_2(\mathbf{v}_n) \\ &= L_2(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) \\ &= L_2(\mathbf{v}). \end{aligned}$$

34. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be in $L^{-1}(W_1)$ and let c be a scalar. Then $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$ is in W_1 since $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ are in W_1 and W_1 is a subspace of V. Hence $\mathbf{v}_1 + \mathbf{v}_2$ is in $L^{-1}(W_1)$. Similarly, $L(c\mathbf{v}_1) = cL(\mathbf{v}_1)$ is in W_1 so $c\mathbf{v}_1$ is in $L^{-1}(W_1)$. Hence, $L^{-1}(W_1)$ is a subspace of V.

35. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the natural basis for R^n. Then $O(\mathbf{e}_i) = \mathbf{0}$ for $i = 1, \ldots, n$. Hence the standard matrix representing O is the $n \times n$ zero matrix O.

36. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the natural basis for R^n. Then $I(\mathbf{e}_i) = \mathbf{e}_i$ for $i = 1, \ldots, n$. Hence the standard matrix representing I is the $n \times n$ identity matrix I_n.

37. Suppose there is another matrix B such that $L(\mathbf{x}) = B\mathbf{x}$ for all $\mathbf{x}$ in R^n. Then $L(\mathbf{e}_j) = B\mathbf{e}_j = \mathrm{Col}_j(B)$ for $j = 1, \ldots, n$. But by definition, $L(\mathbf{e}_j)$ is the jth column of A. Hence $\mathrm{Col}_j(B) = \mathrm{Col}_j(A)$ for $j = 1, \ldots, n$ and therefore $B = A$. Thus the matrix A is unique.

38. (a) 71 52 33 47 30 26 84 56 43 99 69 55. (b) CERTAINLY NOT.

Section 6.2, p. 387

2. (a) No. (b) Yes. (c) Yes. (d) No.
   (e) All vectors of the form $\begin{bmatrix} -2a \\ a \end{bmatrix}$, where a is any real number.
   (f) A possible answer is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$.
4. (a) $\{\begin{bmatrix} 0 & 0 \end{bmatrix}\}$. (b) Yes. (c) No.
6. (a) A possible basis for $\ker L$ is $\{1\}$ and $\dim \ker L = 1$.
   (b) A possible basis for range L is $\{2t^3, t^2\}$ and $\dim \operatorname{range} L = 2$.
8. (a) $\{-t^2 + t + 1\}$. (b) $\{t, 1\}$.
10. (a) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \right\}$. (b) $\left\{ \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.
12. (a) Follows at once from Theorem 6.6.
    (b) If L is onto, then range $L = W$ and the result follows from part (a).
14. (a) If L is one-to-one then $\dim \ker L = 0$, so from Theorem 6.6, $\dim V = \dim \operatorname{range} L$. Hence range $L = W$.
    (b) If L is onto, then $W = \operatorname{range} L$, and since $\dim W = \dim V$, then $\dim \ker L = 0$.
15. If $\mathbf{y}$ is in range L, then $\mathbf{y} = L(\mathbf{x}) = A\mathbf{x}$ for some $\mathbf{x}$ in R^m. This means that $\mathbf{y}$ is a linear combination of the columns of A, so $\mathbf{y}$ is in the column space of A. Conversely, if $\mathbf{y}$ is in the column space of A, then $\mathbf{y} = A\mathbf{x}$, so $\mathbf{y} = L(\mathbf{x})$ and $\mathbf{y}$ is in range L.
16. (a) A possible basis for $\ker L$ is $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$; $\dim \ker L = 2$.
    (b) A possible basis for range L is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$; $\dim \operatorname{range} L = 3$.
18. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. If L is invertible then L is one-to-one: from Theorem 6.7 it follows that $T = \{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$ is linearly independent. Since $\dim W = \dim V = n$, T is a basis for W. Conversely, let the image of a basis for V under L be a basis for W. Let $\mathbf{v} \neq \mathbf{0}_V$ be any vector in V. Then there exists a basis for V including $\mathbf{v}$ (Theorem 4.11). From the hypothesis we conclude that $L(\mathbf{v}) \neq \mathbf{0}_W$. Hence, $\ker L = \{\mathbf{0}_V\}$ and L is one-to-one. From Corollary 6.2 it follows that L is onto. Hence, L is invertible.
19. (a) Range L is spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Since this set of vectors is linearly independent, it is a basis for range L. Hence $L\colon R^3 \to R^3$ is one-to-one and onto.
    (b) $\begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$.

20. If S is linearly dependent then $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}_V$, where $a_1, a_2, \ldots, a_n$ are not all 0. Then

$$a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2) + \cdots + a_nL(\mathbf{v}_n) = L(\mathbf{0}_V) = \mathbf{0}_W,$$

which gives the contradiction that T is linearly dependent. The converse is false: let $L\colon V \to W$ be defined by $L(\mathbf{v}) = \mathbf{0}_W$.

22. A possible answer is $L\left(\begin{bmatrix} u_1 & u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 3u_2 & -u_1 + u_2 & 2u_1 - u_2 \end{bmatrix}$.

23. (a) L is one-to-one and onto. (b) $\begin{bmatrix} 2u_1 - u_3 \\ -2u_1 - u_2 + 2u_3 \\ u_1 + u_2 - u_3 \end{bmatrix}$.

24. If L is one-to-one, then $\dim V = \dim\ker L + \dim\operatorname{range} L = \dim\operatorname{range} L$. Conversely, if $\dim\operatorname{range} L = \dim V$, then $\dim\ker L = 0$.

26. (a) 7; (b) 5.

28. (a) Let $a = 0$, $b = 1$. Let

$$f(x) = \begin{cases} 0 & \text{for } x \neq \frac{1}{2} \\ 1 & \text{for } x = \frac{1}{2}. \end{cases}$$

Then $L(f) = \int_0^1 f(x)\,dx = 0 = L(0)$ so L is not one-to-one.

(b) Let $a = 0$, $b = 1$. For any real number c, let $f(x) = c$ (constant). Then $L(f) = \int_0^1 c\,dx = c$. Thus L is onto.

29. Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $L(\mathbf{x}) = \mathbf{b}$. We show that $\mathbf{x}_1 - \mathbf{x}_2$ is in $\ker L$:

$$L(\mathbf{x}_1 - \mathbf{x}_2) = L(\mathbf{x}_1) - L(\mathbf{x}_2) = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

30. Let $L\colon R^n \to R^m$ be defined by $L(\mathbf{x}) = A\mathbf{x}$, where A is $m \times n$. Suppose that L is onto. Then $\dim\operatorname{range} L = m$. By Theorem 6.6, $\dim\ker L = n - m$. Recall that $\ker L =$ null space of A, so nullity of $A = n - m$. By Theorem 4.19, $\operatorname{rank} A = n -$ nullity of $A = n - (n - m) = m$. Conversely, suppose $\operatorname{rank} A = m$. Then nullity $A = n - m$, so $\dim\ker L = n - m$. Then $\dim\operatorname{range} L = n - \dim\ker L = n - (n - m) = m$. Hence L is onto.

31. From Theorem 6.6, we have $\dim\ker L + \dim\operatorname{range} L = \dim V$.

(a) If L is one-to-one, then $\ker L = \{\mathbf{0}\}$, so $\dim\ker L = 0$. Hence $\dim\operatorname{range} L = \dim V = \dim W$ so L is onto.

(b) If L is onto, then $\operatorname{range} L = W$, so $\dim\operatorname{range} L = \dim W = \dim V$. Hence $\dim\ker L = 0$ and L is one-to-one.

Section 6.3, p. 397

2. (a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 3 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 2 & 0 & 2 \end{bmatrix}$.

4. $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$.

6. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

8. (a) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$. (b) $\begin{bmatrix} 4 & 3 & 0 & 3 \\ -6 & -5 & -4 & -3 \\ 3 & 3 & 7 & 0 \\ 8 & 6 & 4 & 4 \end{bmatrix}$. (c) $\begin{bmatrix} 0 & 3 & 0 & 4 \\ -2 & -3 & -2 & -4 \\ 3 & 0 & 4 & 0 \\ 2 & 4 & 2 & 6 \end{bmatrix}$. (d) $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 3 & 7 & 0 \\ 4 & 3 & 0 & 3 \end{bmatrix}$.

10. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$. (c) $-3t^2 + 3t + 3$.

12. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be an ordered basis for U and $T = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_n\}$ an ordered basis for V (Theorem 4.11). Now $L(\mathbf{v}_j)$ for $j = 1, 2, \dots, m$ is a vector in U, so $L(\mathbf{v}_j)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$. Thus $L(\mathbf{v}_j) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m + 0\mathbf{v}_{m+1} + \cdots + 0\mathbf{v}_n$. Hence,

$$\big[L(\mathbf{v}_j)\big]_T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

14. (a) $\begin{bmatrix} 5 \\ 13 \end{bmatrix}$. (b) $\begin{bmatrix} -5 \\ -3 \end{bmatrix}$. (c) $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$. (d) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. (e) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

15. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an ordered basis for V and $T = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ an ordered basis for W. Now $O(\mathbf{v}_j) = \mathbf{0}_W$ for $j = 1, 2, \dots, n$, so

$$\big[O(\mathbf{v}_j)\big]_T = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

16. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an ordered basis for V. Then $I(\mathbf{v}_j) = \mathbf{v}_j$ for $j = 1, 2, \dots, n$, so

$$\big[I(\mathbf{v}_j)\big]_S = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow j\text{th row.} \\ \\ \\ \\ \end{array}$$

18. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

20. (a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. (d) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

21. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for V. Then $L(\mathbf{v}_i) = c\mathbf{v}_i$. Hence

$$\big[L(\mathbf{v}_i)\big]_S = \begin{bmatrix} 0 \\ \vdots \\ c \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow i\text{th row.} \\ \\ \\ \\ \end{matrix}$$

Thus, the matrix $\begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & c \end{bmatrix} = cI_n$ represents L with respect to S.

22. (a) $\big[L(\mathbf{v}_1)\big]_T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\big[L(\mathbf{v}_2)\big]_T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\big[L(\mathbf{v}_3)\big]_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) $L(\mathbf{v}_1) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $L(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, and $L(\mathbf{v}_3) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$.

23. Let $I\colon V \to V$ be the identity operator defined by $I(\mathbf{v}) = \mathbf{v}$ for $\mathbf{v}$ in V. The matrix A of I with respect to S and T is obtained as follows. The jth column of A is $\big[I(\mathbf{v}_j)\big]_T = \big[\mathbf{v}_j\big]_T$, so as defined in Section 3.7, A is the transition matrix $P_{T \leftarrow S}$ from the S-basis to the T-basis.

Section 6.4, p. 405

1. (a) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V and c_1 and c_2 scalars. Then

$$\begin{aligned}
(L_1 \boxplus L_2)(c_1\mathbf{u} + c_2\mathbf{v}) &= L_1(c_1\mathbf{u} + c_2\mathbf{v}) + L_2(c_1\mathbf{u} + c_2\mathbf{v}) \\
&\qquad \text{(from Definition 6.5)} \\
&= c_1L_1(\mathbf{u}) + c_2L_1(\mathbf{v}) + c_1L_2(\mathbf{u}) + c_2L_2(\mathbf{v}) \\
&\qquad \text{(since } L_1 \text{ and } L_2 \text{ are linear transformations)} \\
&= c_1(L_1(\mathbf{u}) + L_2(\mathbf{u})) + c_2(L_1(\mathbf{v}) + L_2(\mathbf{v})) \\
&\qquad \text{(using properties of vector operations since the images are in } W\text{)} \\
&= c_1(L_1 \boxplus L_2)(\mathbf{u}) + c_2(L_1 \boxplus L_2)(\mathbf{v}) \\
&\qquad \text{(from Definition 6.5)}
\end{aligned}$$

Thus by Exercise 4 in Section 6.1, $L_1 \boxplus L_2$ is a linear transformation.

(b) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V and k_1 and k_2 be scalars. Then

$$\begin{aligned}
(c \boxdot L)(k_1\mathbf{u} + k_2\mathbf{v}) &= cL(k_1\mathbf{u} + k_2\mathbf{v}) \\
&\qquad \text{(from Definition 6.5)} \\
&= c(k_1L(\mathbf{u}) + k_2L(\mathbf{v})) \\
&\qquad \text{(since } L \text{ is a linear transformation)} \\
&= ck_1L(\mathbf{u}) + ck_2L(\mathbf{v}) \\
&\qquad \text{(using properties of vector operations since the images are in } W\text{)} \\
&= k_1cL(\mathbf{u}) + k_2cL(\mathbf{v}) \\
&\qquad \text{(using properties of vector operations)} \\
&= k_1(c \boxdot L)(\mathbf{u}) + k_2(c \boxdot L)(\mathbf{v}) \\
&\qquad \text{(by Definition 6.5)}
\end{aligned}$$

(c) Let $S = \{\mathbf{v}_1, \mathbf{v}_2. \dots, \mathbf{v}_n\}$. Then

$$A = \Big[\ \big[L(\mathbf{v}_1\big]_T \quad \big[L(\mathbf{v}_2)\big]_T \quad \cdots \quad \big[L(\mathbf{v}_n)\big]_T\ \Big].$$

The matrix representing $c \boxdot L$ is given by

$$\begin{aligned}
&\Big[\ \big[L(\mathbf{v}_1\big]_T \quad \big[L(\mathbf{v}_2)\big]_T \quad \cdots \quad \big[L(\mathbf{v}_n)\big]_T\ \Big] \\
&\qquad = \Big[\ \big[c \boxdot L(\mathbf{v}_1)\big]_T \quad \big[c \boxdot L(\mathbf{v}_2)\big]_T \quad \cdots \quad \big[c \boxdot L(\mathbf{v}_n)\big]_T\ \Big] \\
&\qquad = \Big[\ \big[cL(\mathbf{v}_1)\big]_T \quad \big[cL(\mathbf{v}_2)\big]_T \quad \cdots \quad \big[cL(\mathbf{v}_n)\big]_T\ \Big] \\
&\qquad\qquad \text{(by Definition 6.5)} \\
&\qquad = \Big[\ c\big[L(\mathbf{v}_1)\big]_T \quad c\big[L(\mathbf{v}_2)\big]_T \quad \cdots \quad c\big[L(\mathbf{v}_n)\big]_T\ \Big] \\
&\qquad\qquad \text{(by properties of coordinates)} \\
&\qquad = c\Big[\ \big[L(\mathbf{v}_1)\big]_T \quad \big[L(\mathbf{v}_2)\big]_T \quad \cdots \quad \big[L(\mathbf{v}_n)\big]_T\ \Big] = cA \\
&\qquad\qquad \text{(by matrix algebra)}
\end{aligned}$$

2. (a) $(O \boxplus L)(\mathbf{u}) = O(\mathbf{u}) + L(\mathbf{u}) = L(\mathbf{u})$ for any $\mathbf{u}$ in V.

   (b) For any $\mathbf{u}$ in V, we have

$$[L \boxplus ((-1) \boxdot L)](\mathbf{u}) = L(\mathbf{u}) + (-1)L(\mathbf{u}) = \mathbf{0} = O(\mathbf{u}).$$

4. Let L_1 and L_2 be linear transformations of V into W. Then $L_1 \boxplus L_2$ and $c \boxdot L_1$ are linear transformations by Exercise 1 (a) and (b). We must now verify that the eight properties of Definition 4.4 are satisfied. For example, if $\mathbf{v}$ is any vector in V, then

$$(L_1 \boxplus L_2)(\mathbf{v}) = L_1(\mathbf{v}) + L_2(\mathbf{v}) = L_2(\mathbf{v}) + L_1(\mathbf{v}) = (L_2 \boxplus L_1)(\mathbf{v}).$$

   Therefore, $L_1 \boxplus L_2 = L_2 \boxplus L_1$. The remaining seven properties are verified in a similar manner.

6. $(L_2 \circ L_1)(a\mathbf{u} + b\mathbf{v}) = L_2(L_1(a\mathbf{u} + b\mathbf{v})) = L_2(aL_1(\mathbf{u}) + bL_1(\mathbf{v}))$
   $= aL_2(L_1(\mathbf{u})) + bL_2(L_1(\mathbf{v})) = a(L_2 \circ L_1)(\mathbf{u}) + b(L_2 \circ L_1)(\mathbf{v})$.

8. (a) $\begin{bmatrix} -3u_1 - 5u_2 - 2u_3 & 4u_1 + 7u_2 + 4u_3 & 11u_1 + 3u_2 + 10u_3 \end{bmatrix}$.

   (b) $\begin{bmatrix} 8u_1 + 4u_2 + 4u_3 & -3u_1 + 2u_2 + 3u_3 & u_1 + 5u_2 + 4u_3 \end{bmatrix}$.

(c) $\begin{bmatrix} -3 & -5 & -2 \\ 4 & 7 & 4 \\ 11 & 3 & 10 \end{bmatrix}$. (d) $\begin{bmatrix} 8 & 4 & 4 \\ -3 & 2 & 3 \\ 1 & 5 & 4 \end{bmatrix}$.

10. Consider $u_1 L_1 + u_2 L_2 + u_3 L_3 = O$. Then

$$\begin{aligned}(u_1 L_1 + u_2 L_2 + u_3 L_3)\left(\begin{bmatrix}1 & 0 & 0\end{bmatrix}\right) &= O\left(\begin{bmatrix}1 & 0 & 0\end{bmatrix}\right) \\ &= \begin{bmatrix}0 & 0\end{bmatrix} \\ &= u_1 \begin{bmatrix}1 & 1\end{bmatrix} + u_2 \begin{bmatrix}1 & 0\end{bmatrix} + u_3 \begin{bmatrix}1 & 0\end{bmatrix} \\ &= \begin{bmatrix}u_1 + u_2 + u_3 & u_1\end{bmatrix}.\end{aligned}$$

Thus, $u_1 = 0$. Also,

$$(u_1 L_1 + u_2 L_2 + u_3 L_3)\left(\begin{bmatrix}0 & 1 & 0\end{bmatrix}\right) = O\left(\begin{bmatrix}0 & 1 & 0\end{bmatrix}\right) = \begin{bmatrix}0 & 0\end{bmatrix} = \begin{bmatrix}u_1 - u_2 & u_3\end{bmatrix}.$$

Thus $u_2 = u_3 = 0$.

12. (a) 4. (b) 16. (c) 6.

13. (a) Verify that $L(a\mathbf{u} + b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v})$.

(b) $L(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \cdots + a_{mj}\mathbf{w}_m$, so $\left[L(\mathbf{v}_j)\right]_T = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ = the jth column of A. Hence A represents L with respect to S and T.

14. (a) $L(\mathbf{e}_1) = \begin{bmatrix}1 \\ 3\end{bmatrix}$, $L(\mathbf{e}_2) = \begin{bmatrix}2 \\ 4\end{bmatrix}$, $L(\mathbf{e}_3) = \begin{bmatrix}-2 \\ -1\end{bmatrix}$.

(b) $\begin{bmatrix}u_1 + 2u_2 - 2u_3 \\ 3u_1 + 4u_2 - u_3\end{bmatrix}$.

(c) $\begin{bmatrix}-1 \\ 8\end{bmatrix}$.

16. Possible answer: $L_1\left(\begin{bmatrix}u_1 \\ u_2\end{bmatrix}\right) = \begin{bmatrix}u_1 \\ -u_2\end{bmatrix}$ and $L_2\left(\begin{bmatrix}u_1 \\ u_2\end{bmatrix}\right) = \begin{bmatrix}u_2 \\ u_1\end{bmatrix}$.

18. Possible answers: $L\left(\begin{bmatrix}u_1 \\ u_2\end{bmatrix}\right) = \begin{bmatrix}u_1 - u_2 \\ u_1 - u_2\end{bmatrix}$; $L\left(\begin{bmatrix}u_1 \\ u_2\end{bmatrix}\right) = \begin{bmatrix}0 \\ u_1\end{bmatrix}$.

20. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

22. $\begin{bmatrix} -\frac{9}{2} & -6 & 2 \\ \frac{1}{2} & 1 & 0 \\ \frac{5}{2} & 3 & -1 \end{bmatrix}$.

23. From Theorem 6.11, it follows directly that A^2 represents $L^2 = L \circ L$. Now Theorem 6.11 implies that A^3 represents $L^3 = L \circ L^2$. We continue this argument as long as necessary. A more formal proof can be given using induction.

24. $\begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{3}{10} & -\frac{2}{5} \end{bmatrix}$.

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1. (a) $A = I_n^{-1} A I_n$.
   (b) If $B = P^{-1}AP$ then $A = PBP^{-1}$. Let $P^{-1} = Q$ so $A = Q^{-1}BQ$.
   (c) If $B = P^{-1}AP$ and $C = Q^{-1}BQ$, then $C = Q^{-1}P^{-1}APQ$ and letting $M = PQ$ we get $C = M^{-1}AM$.

2. (a) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (d) $\begin{bmatrix} 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 3 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. (e) 3.

4. $P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix}$.

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 3 & 0 & 4 \\ -2 & -3 & -2 & -4 \\ 3 & 0 & 4 & 0 \\ 2 & 4 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 & 3 \\ -6 & -5 & -4 & -3 \\ 3 & 3 & 7 & 0 \\ 8 & 6 & 4 & 4 \end{bmatrix}.$$

6. If $B = P^{-1}AP$, then $B^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$. Thus, A^2 and B^2 are similar, etc.

7. If $B = P^{-1}AP$, then $B^T = P^T A^T (P^{-1})^T$. Let $Q = (P^{-1})^T$, so $B^T = Q^{-1}A^TQ$.

8. If $B = P^{-1}AP$, then $\mathrm{Tr}(B) = \mathrm{Tr}(P^{-1}AP) = \mathrm{Tr}(APP^{-1}) = \mathrm{Tr}(AI_n) = \mathrm{Tr}(A)$.

10. Possible answer: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

11. (a) If $B = P^{-1}AP$ and A is nonsingular then B is nonsingular.
    (b) If $B = P^{-1}AP$ then $B^{-1} = P^{-1}A^{-1}P$.

12. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

14. $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$B = Q^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}.$$

16. A and O are similar if and only if $A = P^{-1}OP = O$ for a nonsingular matrix P.

17. Let $B = P^{-1}AP$. Then $\det(B) = \det(P^{-1}AP) = \det(P)^{-1}\det(A)\det(P) = \det(A)$.

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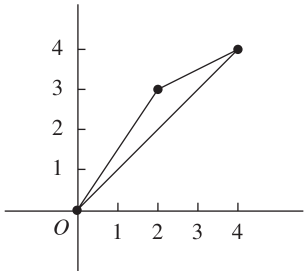
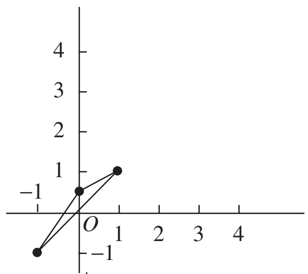
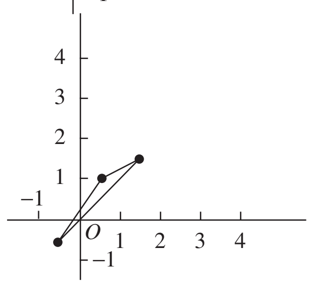
2. (a) 

(b) $M = \begin{bmatrix} \frac{1}{2} & 0 & -1 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. 

(d) $Q = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$.

(e) $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$. 

(f) No. The images are not the same since the matrices M and Q are different.

4. (a) $M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Yes, compute P^{-1}; $P^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

6. $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. The images will be the same since $AB = BA$.

8. The original triangle is reflected about the x-axis and then dilated (scaled) by a factor of 2. Thus the matrix M that performs these operations is given by

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the two matrices are diagonal and diagonal matrices commute under multiplication, hence the order of the operations is not relevant.

10. Here there are various ways to proceed depending on how one views the mapping.

Solution #1: The original semicircle is dilated by a factor of 2. The point at $(1, 1)$ now corresponds to a point at $(2, 2)$. Next we translate the point $(2, 2)$ to the point $(-6, 2)$. In order to translate point

$(2,2)$ to $(-6,2)$ we add -8 to the x-coordinate and 0 to the y-coordinate. Thus the matrix M that performs these operations is given by

$$M = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -8 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution #2: The original semicircle is translated so that the point $(1,1)$ corresponds to point $(-3,1)$. In order to translate point $(1,1)$ to $(-3,1)$ we add -4 to the x-coordinate and 0 to the y-coordinate. Next we perform a scaling by a factor of 2. Thus the matrix M that performs these operations is given by

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -8 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the matrix of the composite transformation is the same, yet the matrices for the individual steps differ.

12. The image can be obtained by first translating the semicircle to the origin and then rotating it $-45°$. Using this procedure the corresponding matrix is

$$M = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\sqrt{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

14. (a) Since we are translating down the y-axis, only the y coordinates of the vertices of the triangle change. The matrix for this sweep is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s_{j+1}10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) If we translate and then rotate for each step the composition of the operations is given by the matrix product

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s_{j+1}10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(s_{j+1}\pi/4) & 0 & \sin(s_{j+1}\pi/4) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(s_{j+1}\pi/4) & 0 & \cos(s_{j+1}\pi/4) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos(s_{j+1}\pi/4) & 0 & \sin(s_{j+1}\pi/4) & 0 \\ 0 & 1 & 0 & s_{j+1}10 \\ -\sin(s_{j+1}\pi/4) & 0 & \cos(s_{j+1}\pi/4) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) Take the composition of the sweep matrix from part (a) with a scaling by $\frac{1}{2}$ in the z-direction. In the scaling matrix we must write the parameterization so it decreases from 1 to $\frac{1}{2}$, hence we use $1 - s_{j+1}\left(\frac{1}{2}\right)$. We obtain the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s_{j+1}10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 - s_{j+1}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s_{j+1}10 \\ 0 & 0 & 1 - s_{j+1}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Supplementary Exercises for Chapter 6, p. 430

1. Let A and B belong to M_{nm} and let c be a scalar. From Exercise 43 in Section 1.3 we have that $\text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B)$ and $\text{Tr}(cA) = c\,\text{Tr}(A)$. Thus Definition 6.1 is satisfied and it follows that Tr is a linear transformation.

2. Let A and B belong to M_{nm} and let c be a scalar. Then $L(A+B) = (A+B)^T = A^T + B^T = L(A) + L(B)$ and $L(cA) = (cA)^T = cA^T = cL(A)$, so L is a linear transformation.

4. (a) $\begin{bmatrix} 3 & 4 & 8 \end{bmatrix}$.
   (b) $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.
   (c) $\frac{1}{3}\begin{bmatrix} 2u_1 + u_2 + u_3 & 4u_1 - 4u_2 + 2u_3 & 7u_1 - 4u_2 + 2u_3 \end{bmatrix}$.

6. (a) No. (b) Yes. (c) Yes. (d) No. (e) $-t^2 - t + 1$ (f) t^2, t.

8. (a) $\ker L = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$; it has no basis. (b) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

10. A possible basis consists of any nonzero constant function.

12. (a) A possible basis is $\left\{ t - \frac{1}{2} \right\}$.
    (b) A possible basis is $\{1\}$.
    (c) $\dim \ker L + \dim \operatorname{range} L = 1 + 1 = 2 = \dim P_1$.

14. (a) $L(p_1(t)) = 3t - 3$, $L(p_2(t)) = -t + 8$.
    (b) $\left[L(p_1(t))\right]_S = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\left[L(p_2(t))\right]_S = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.
    (c) $\frac{7}{3}(-t + 5)$.

16. Let $\mathbf{u}$ be any vector in R^n and assume that $\|L(\mathbf{u})\| = \|\mathbf{u}\|$. From Theorem 6.9, if we let S be the standard basis for R^n then there exists an $n \times n$ matrix A such that $L(\mathbf{u}) = A\mathbf{u}$. Then

$$\|L(\mathbf{u})\|^2 = (L(\mathbf{u}), L(\mathbf{u})) = (A\mathbf{u}, A\mathbf{u}) = (\mathbf{u}, A^T A\mathbf{u})$$

by Equation (3) of Section 5.3,, and it then follows that $(\mathbf{u}, \mathbf{u}) = (\mathbf{u}, A^T A\mathbf{u})$. Since $A^T A$ is symmetric, Supplementary Exercise 17 of Chapter 5 implies that $A^T A = I_n$. It follows that for $\mathbf{v}$, $\mathbf{w}$ any vectors in R^n,

$$(L(\mathbf{u}), L(\mathbf{v})) = (A\mathbf{u}, A\mathbf{v}) = (\mathbf{u}, A^T A\mathbf{v}) = (\mathbf{u}, \mathbf{v}).$$

Conversely, assume that $(L(\mathbf{u}), L(\mathbf{v})) = (\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in R^n. Then $\|L(\mathbf{u})\|^2 = (L(\mathbf{u}), L(\mathbf{u})) = (\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2$, so $\|L(\mathbf{u})\| = \|\mathbf{u}\|$.

17. Assume that $(L_1 + L_2)^2 = L_1^2 + 2L_1 \circ L_2 + L_2^2$. Then

$$L_1^2 + L_1 \circ L_2 + L_2 \circ L_1 + L_2^2 = L_1^2 + 2L_1 \circ L_2 + L_2^2,$$

and simplifying gives $L_1 \circ L_2 = L_2 \circ L_1$. The steps are reversible.

18. If $(L(\mathbf{u}), L(\mathbf{v})) = (\mathbf{u}, \mathbf{v})$ then

$$\cos\theta = \frac{(L(\mathbf{u}), L(\mathbf{v}))}{\|L(\mathbf{u})\|\,\|L(\mathbf{v})\|} = \frac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

where θ is the angle between $L(\mathbf{u})$ and $L(\mathbf{v})$. Thus θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

19. (a) Suppose that $L(\mathbf{v}) = \mathbf{0}$. Then $\mathbf{0} = (0, 0) = (L(\mathbf{v}), L(\mathbf{v})) = (\mathbf{v}, \mathbf{v})$. But then from the definition of an inner product, $\mathbf{v} = \mathbf{0}$. Hence $\ker L = \{\mathbf{0}\}$.

(b) See the proof of Exercise 16.

20. Let $\mathbf{w}$ be any vector in range L. Then there exists a vector $\mathbf{v}$ in V such that $L(\mathbf{v}) = \mathbf{w}$. Next there exists scalars $c_1, \ldots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$. Thus

$$\mathbf{w} = L(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) = c_1L(\mathbf{v}_1) + \cdots + c_kL(\mathbf{v}_k).$$

Hence $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_k)\}$ spans range L.

21. (a) We use Exercise 4 in Section 6.1 to show that L is a linear transformation. Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

be vectors in R^n and let r and s be scalars. Then

$$\begin{aligned} L(r\mathbf{u} + s\mathbf{v}) &= L\left(r\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + s\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) = L\left(\begin{bmatrix} ru_1 + sv_1 \\ ru_2 + sv_2 \\ \vdots \\ ru_n + sv_n \end{bmatrix} \right) \\ &= (ru_1 + sv_1)\mathbf{v}_1 + (ru_2 + sv_2)\mathbf{v}_2 + \cdots + (ru_n + sv_n)\mathbf{v}_n \\ &= r(u_1\mathbf{v}_1 + u_2\mathbf{v}_2 + \cdots + u_n\mathbf{v}_n) + s(v_1\mathbf{v}_1 + v_2\mathbf{v}_2 + \cdots + v_n\mathbf{v}_n) \\ &= rL(\mathbf{u}) + sL(\mathbf{v}) \end{aligned}$$

Therefore L is a linear transformation.

(b) We show that $\ker L = \{\mathbf{0}_V\}$. Let $\mathbf{v}$ be in the kernel of L. Then $L(\mathbf{v}) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_n\mathbf{v}_n = \mathbf{0}$. Since the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_n$ form a basis for V, they are linearly independent. Therefore $a_1 = 0$, $a_2 = 0$, $\ldots$, $a_n = 0$. Hence $\mathbf{v} = \mathbf{0}$. Therefore $\ker L = \{\mathbf{0}\}$ and hence L is one-to-one by Theorem 6.4.

(c) Since both R^n and V have dimension n, it follows from Corollary 6.2 that L is onto.

22. By Theorem 6.10, $\dim V^* = n \cdot 1 = n$, so $\dim V^* = \dim V$. This implies that V and V^* are isomorphic vector spaces.

23. We have $BA = A^{-1}(AB)A$, so AB and BA are similar.

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True or False

1. True.	2. False.	3. True.	4. False.	5. False.
6. True.	7. True.	8. True.	9. True.	10. False.
11. True.	12. False.			

Quiz

1. Yes. 2. (b) $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$.

3. (a) Possible answer: $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 3 & -1 \end{bmatrix}$. (b) No.

4. $\begin{bmatrix} -4 \\ 3 \\ 4 \end{bmatrix}$. 5. $\begin{bmatrix} 0 & -1 \\ 3 & 5 \end{bmatrix}$.

6. (a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. (d) $\begin{bmatrix} -1 & 1 \\ 2 & 0 \\ -1 & -1 \end{bmatrix}$.

Chapter 7

Eigenvalues and Eigenvectors

Section 7.1, p. 450

2. The characteristic polynomial is $\lambda^2 - 1$, so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. Associated eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

4. The eigenvalues of L are $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = 3$. Associated eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}$.

6. (a) $p(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$. The eigenvalues and associated eigenvectors are:

$$\lambda_1 = 0; \qquad \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2; \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) $p(\lambda) = \lambda^3 - 2\lambda^2 - 5\lambda + 6 = (\lambda + 2)(\lambda - 1)(\lambda - 3)$. The eigenvalues and associated eigenvectors are

$$\lambda_1 = -2; \qquad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1; \qquad \mathbf{x}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$$

$$\lambda_3 = 3; \qquad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$$

(c) $p(\lambda) = \lambda^3$. The eigenvalues and associated eigenvectors are

$$\lambda_1 = \lambda_2 = \lambda_3 = 0; \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(d) $p(\lambda) = \lambda^3 - 5\lambda^2 + 2\lambda + 8 = (\lambda+1)(\lambda-2)(\lambda-4)$. The eigenvalues and associated eigenvectors are

$$\lambda_1 = -1; \qquad \mathbf{x}_1 = \begin{bmatrix} -8 \\ 10 \\ 7 \end{bmatrix}$$

$$\lambda_2 = 2; \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 4; \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

8. (a) $p(\lambda) = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$. The eigenvalues and associated eigenvectors are:

$$\lambda_1 = 2; \qquad \mathbf{x}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3; \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(b) $p(\lambda) = \lambda^2 + 9$. No eigenvalues or eigenvectors.

(c) $p(\lambda) = \lambda^3 - 15\lambda^2 + 72\lambda - 108 = (\lambda - 3)(\lambda - 6)^2$. The eigenvalues and associated eigenvectors are:

$$\lambda_1 = 3; \qquad \mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \lambda_3 = 6; \qquad \mathbf{x}_2 = \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(d) $p(\lambda) = \lambda^3 + \lambda = \lambda(\lambda^2 + 1)$. The eigenvalues and associated eigenvectors are:

$$\lambda_1 = 0; \qquad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

10. (a) $p(\lambda) = \lambda^2 + \lambda + 1 - i = (\lambda - i)(\lambda + 1 + i)$. The eigenvalues and associated eigenvectors are:

$$\lambda_1 = i; \qquad \mathbf{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 - i; \qquad \mathbf{x}_2 = \begin{bmatrix} -1 - i \\ 1 \end{bmatrix}$$

(b) $p(\lambda) = (\lambda - 1)(\lambda^2 - 2i\lambda - 2) = (\lambda - 1)[\lambda - (1 + i)][\lambda - (-1 + i)]$. The eigenvalues and associated

eigenvectors are:

$$\lambda_1 = 1+i; \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1+i; \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 1; \qquad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) $p(\lambda) = \lambda^3 + \lambda = \lambda(\lambda + i)(\lambda - i)$. The eigenvalues and associated eigenvectors are:

$$\lambda_1 = 0; \qquad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = i; \qquad \mathbf{x}_2 = \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}$$

$$\lambda_3 = -i; \qquad \mathbf{x}_1 = \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}$$

(d) $p(\lambda) = \lambda^2(\lambda-1)+9(\lambda-1) = (\lambda-1)(\lambda-3i)(\lambda+3i)$. The eigenvalues and associated eigenvectors are:

$$\lambda_1 = 1; \qquad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 3i; \qquad \mathbf{x}_2 = \begin{bmatrix} 3i \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -3i; \qquad \mathbf{x}_1 = \begin{bmatrix} -3i \\ 0 \\ 1 \end{bmatrix}$$

11. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an $n \times n$ upper triangular matrix, that is, $a_{ij} = 0$ for $i > j$. Then the characteristic polynomial of A is

$$p(\lambda) = \det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ 0 & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda - a_{nn} \end{vmatrix} = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}),$$

which we obtain by expanding along the cofactors of the first column repeatedly. Thus the eigenvalues of A are $a_{11}, \ldots, a_{nn}$, which are the elements on the main diagonal of A. A similar proof shows the same result if A is lower triangular.

12. We prove that A and A^T have the same characteristic polynomial. Thus

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I_n - A) = \det((\lambda I_n - A)^T) = \det(\lambda I_n^T - A^T) \\ &= \det(\lambda I_n - A^T) = p_{A^T}(\lambda). \end{aligned}$$

Associated eigenvectors need not be the same for A and A^T. As a counterexample, consider the matrix in Exercise 7(c) for $\lambda_2 = 2$.

14. Let V be an n-dimensional vector space and $L : V \to V$ be a linear operator. Let λ be an eigenvalue of L and W the subset of V consisting of the zero vector $\mathbf{0}_V$, and all the eigenvectors of L associated with λ. To show that W is a subspace of V, let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors of L corresponding to λ and let c_1 and c_2 be scalars. Then $L(\mathbf{u}) = \lambda\mathbf{u}$ and $L(\mathbf{v}) = \lambda\mathbf{v}$. Therefore

$$L(c_1\mathbf{u} + c_2\mathbf{v}) = c_1 L(\mathbf{u}) + c_2 L(\mathbf{v}) = c_1\lambda\mathbf{u} + c_2\lambda\mathbf{v} = \lambda(c_1\mathbf{u} + c_2\mathbf{v}).$$

Thus $c_1\mathbf{u}+c_2\mathbf{v}$ is an eigenvector of L with eigenvalue λ. Hence W is closed with respect to addition and scalar multiplication. Since technically an eigenvector is never zero we had to explicitly state that $\mathbf{0}_V$ was in W since scalars c_1 and c_2 could be zero or $c_1 = -c_2$ and $\mathbf{u} = \mathbf{v}$ making the linear combination $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}_V$. It follows that W is a subspace of V.

15. We use Exercise 14 as follows. Let $L : R^n \to R^n$ be defined by $L(\mathbf{x}) = A\mathbf{x}$. Then we saw in Chapter 4 that L is a linear transformation and matrix A represents this transformation. Hence Exercise 14 implies that all the eigenvectors of A with associated eigenvalue λ, together with the zero vector, form a subspace of V.

16. To be a subspace, the subset must be closed under scalar multiplication. Thus, if $\mathbf{x}$ is any eigenvector, then $0\mathbf{x} = \mathbf{0}$ must be in the subset. Since the zero vector is not an eigenvector, we must include it in the subset of eigenvectors so that the subset is a subspace.

18. (a) $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

20. (a) Possible answer: $\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$. (b) Possible answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

21. If λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$, then $A\mathbf{x} = \lambda\mathbf{x}$. This implies that $A(A\mathbf{x}) = A(\lambda\mathbf{x})$, so that $A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$. Thus, λ^2 is an eigenvalue of A^2 with associated eigenvector $\mathbf{x}$. Repeat k times.

22. Let $A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 5 & -4 \\ -1 & 8 \end{bmatrix}$. The characteristic polynomial of A^2 is

$$\det(\lambda I_2 - A^2) = \begin{vmatrix} \lambda - 5 & 4 \\ 1 & \lambda - 8 \end{vmatrix} = (\lambda - 5)(\lambda - 8) - 4 = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9).$$

Thus the eigenvalues of A^2 are $\lambda_1 = 9$ and $\lambda_2 = 4$ which are the squares of the eigenvalues of matrix A. (See Exercise 8(a).) To find an eigenvector corresponding to $\lambda_1 = 9$ we solve the homogeneous linear system

$$(9I_2 - A^2)\mathbf{x} = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Row reducing the coefficient matrix we have the equivalent linear system

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

whose solution is $x_1 = r$, $x_2 = -r$, or in matrix form

$$\mathbf{x} = r \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus $\lambda_1 = 9$ has eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

To find eigenvectors corresponding to $\lambda_2 = 4$ we solve the homogeneous linear system

$$(4I_2 - A^2)\mathbf{x} = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Row reducing the coefficient matrix we have the equivalent linear system

$$\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

whose solution is $x_1 = 4r$, $x_2 = r$, or in matrix form

$$\mathbf{x} = r \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Thus $\lambda_2 = 4$ has eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

We note that the eigenvectors of A^2 are eigenvectors of A corresponding to the square of the eigenvalues of A.

23. If A is nilpotent then $A^k = O$ for some positive integer k. If λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$, then by Exercise 21 we have $O = A^k\mathbf{x} = \lambda^k\mathbf{x}$. Since $\mathbf{x} \neq \mathbf{0}$, $\lambda^k = 0$ so $\lambda = 0$.

24. (a) The characteristic polynomial of A is

$$f(\lambda) = \det(\lambda I_n - A).$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of the characteristic polynomial. Then

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Setting $\lambda = 0$ in each of the preceding expressions for $f(\lambda)$ we have

$$f(0) = \det(-A) = (-1)^n \det(A)$$

and

$$f(0) = (-\lambda_1)(-\lambda_2) \cdots (-\lambda_n) = (-1)^n \lambda_1\lambda_2 \cdots \lambda_n.$$

Equating the expressions for $f(0)$ gives $\det(A) = \lambda_1\lambda_2 \cdots \lambda_n$. That is, $\det(A)$ is the product of the roots of the characteristic polynomial of A.

(b) We use part (a). A is singular if and only if $\det(A) = 0$. Hence $\lambda_1\lambda_2 \cdots \lambda_n = 0$ which is true if and only if some $\lambda_j = 0$. That is, if and only if some eigenvalue of A is zero.

(c) Assume that L is not one-to-one. Then ker L contains a nonzero vector, say $\mathbf{x}$. Then $L(\mathbf{x}) = \mathbf{0}_V = (0)\mathbf{x}$. Hence 0 is an eigenvalue of L. Conversely, assume that 0 is an eigenvalue of L. Then there exists a nonzero vector $\mathbf{x}$ such that $L(\mathbf{x}) = 0\mathbf{x}$. But $0\mathbf{x} = \mathbf{0}_V$, hence ker L contains a nonzero vector so L is not one-to-one.

(d) From Exercise 23, if A is nilpotent then zero is an eigenvalue of A. It follows from part (b) that such a matrix is singular.

25. (a) Since $L(\mathbf{x}) = \lambda\mathbf{x}$ and since L is invertible, we have $\mathbf{x} = L^{-1}(\lambda\mathbf{x}) = \lambda L^{-1}(\mathbf{x})$. Therefore $L^{-1}(\mathbf{x}) = (1/\lambda)\mathbf{x}$. Hence $1/\lambda$ is an eigenvalue of L^{-1} with associated eigenvector $\mathbf{x}$.

(b) Let A be a nonsingular matrix with eigenvalue λ and associated eigenvector $\mathbf{x}$. Then $1/\lambda$ is an eigenvalue of A^{-1} with associated eigenvector $\mathbf{x}$. For if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-1}\mathbf{x} = (1/\lambda)\mathbf{x}$.

26. Suppose there is a vector $\mathbf{x} \neq \mathbf{0}$ in both S_1 and S_2. Then $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{x} = \lambda_2\mathbf{x}$. So $(\lambda_2 - \lambda_1)\mathbf{x} = \mathbf{0}$. Hence $\lambda_1 = \lambda_2$ since $\mathbf{x} \neq \mathbf{0}$, a contradiction. Thus the zero vector is the only vector in both S_1 and S_2.

27. If $A\mathbf{x} = \lambda\mathbf{x}$, then, for any scalar r,

$$(A + rI_n)\mathbf{x} = A\mathbf{x} + r\mathbf{x} = \lambda\mathbf{x} + r\mathbf{x} = (\lambda + r)\mathbf{x}.$$

Thus $\lambda + r$ is an eigenvalue of $A + rI_n$ with associated eigenvector $\mathbf{x}$.

28. Let W be the eigenspace of A with associated eigenvalue λ. Let $\mathbf{w}$ be in W. Then $L(\mathbf{w}) = A\mathbf{w} = \lambda\mathbf{w}$. Therefore $L(\mathbf{w})$ is in W since W is closed under scalar multiplication.

29. (a) $(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x} = (\lambda + \mu)\mathbf{x}$

(b) $(AB)\mathbf{x} = A(B\mathbf{x}) = A(\mu\mathbf{x}) = \mu(A\mathbf{x}) = \mu\lambda\mathbf{x} = (\lambda\mu)\mathbf{x}$

30. (a) The characteristic polynomial is $p(\lambda) = \lambda^3 - \lambda^2 - 24\lambda - 36$. Then

$$p(A) = A^3 - A^2 - 24A - 36I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) The characteristic polynomial is $p(\lambda) = \lambda^3 - 7\lambda + 6$. Then

$$p(A) = A^3 - 7A + 6I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) The characteristic polynomial is $p(\lambda) = \lambda^2 - 7\lambda + 6$. Then

$$p(A) = A^2 - 7A + 6I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

31. Let A be an $n \times n$ nonsingular matrix with characteristic polynomial

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n.$$

By the Cayley-Hamilton Theorem (see Exercise 30)

$$p(A) = A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI_n = O.$$

Multiply the preceding expression by A^{-1} to obtain

$$A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I_n + a_nA^{-1} = O.$$

Rearranging terms we have

$$a_n A^{-1} = -A^{n-1} - a_1 A^{n-2} - \cdots - a_{n-1} I_n.$$

Since A is nonsingular $\det(A) \neq 0$. From the discussion prior to Example 11, $a_n = (-1)^n \det(A)$, so $a_n \neq 0$. Hence we have

$$A^{-1} = -\frac{1}{a_n}\left(A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I_n\right).$$

32. The characteristic polynomial of A is

$$p(\lambda) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - \mathrm{Tr}(A) + \det(A).$$

33. Let A be an $n \times n$ matrix all of whose columns add up to 1 and let $\mathbf{x}$ be the $m \times 1$ matrix

$$\mathbf{x} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Then

$$A^T\mathbf{x} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{x} = 1\mathbf{x}.$$

Therefore $\lambda = 1$ is an eigenvalues of A^T. By Exercise 12, $\lambda = 1$ is an eigenvalue of A.

34. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. Then $a_{kj} = 0$ if $k \neq j$ and $a_{kk} = 1$. We now form $\lambda I_n - A$ and compute the characteristic polynomial of A as $\det(\lambda I_n - A)$ by expanding about the kth row. We obtain $(\lambda - 1)$ times a polynomial of degree $n - 1$. Hence 1 is a root of the characteristic polynomial and is thus an eigenvalue of A.

35. (a) Since $A\mathbf{u} = \mathbf{0} = 0\mathbf{u}$, it follows that 0 is an eigenvalue of A with associated eigenvector $\mathbf{u}$.

    (b) Since $A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$, it follows that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, namely $\mathbf{x} = \mathbf{v}$.

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2. The characteristic polynomial of A is $p(\lambda) = \lambda^2 - 1$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The corresponding vectors in P_1 are

$$\mathbf{x}_1 : p(t) = t - 1; \quad \mathbf{x}_2 : p_2(t) = t + 1.$$

Since the set of eigenvectors $\{t - 1, t + 1\}$ is linearly independent, it is a basis for P_1. Thus P_1 has a basis of eigenvectors of L and hence L is diagonalizable.

4. Yes. Let $S = \{\sin t, \cos t\}$. We first find a matrix A representing L. We use the basis S. We have $L(\sin t) = \cos t$ and $L(\cos t) = -\sin t$. Hence

$$A = \begin{bmatrix} \begin{bmatrix} L(\sin t) \end{bmatrix}_S & \begin{bmatrix} L(\cos t) \end{bmatrix}_S \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We find the eigenvalues and associated eigenvectors of A. The characteristic polynomial of A is

$$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1.$$

This polynomial has roots $\lambda = \pm i$, hence according to Theorem 7.5, L is diagonalizable.

6. (a) Diagonalizable. The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$. The result follows by Theorem 7.5.

   (b) Not diagonalizable. The eigenvalues are $\lambda_1 = \lambda_2 = 1$. Associated eigenvectors are $\mathbf{x}_1 = \mathbf{x}_2 = r\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where r is any nonzero real number.

   (c) Diagonalizable. The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 3$. The result follows by Theorem 7.5.

   (d) Diagonalizable. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 2$. The result follows by Theorem 7.5.

   (e) Not diagonalizable. The eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 3$. Associated eigenvectors are

$$\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 = r\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   where r is any nonzero real number.

8. Let

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}.$$

Then $P^{-1}AP = D$, so

$$A = PDP^{-1} = \frac{1}{3}\begin{bmatrix} -4 & -5 \\ -10 & 1 \end{bmatrix}$$

is a matrix whose eigenvalues and associated eigenvectors are as given.

10. (a) There is no such P. The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = 3$. Associated eigenvectors are

$$\mathbf{x}_1 = \mathbf{x}_2 = r\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

   where r is any nonzero real number, and

$$\mathbf{x}_3 = \begin{bmatrix} -5 \\ -2 \\ 3 \end{bmatrix}.$$

   (b) $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = 3$. Associated eigenvectors are the columns of P.

   (c) $P = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & -6 \\ 1 & 2 & 4 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 4$, $\lambda_2 = -1$, and $\lambda_3 = 1$. Associated eigenvectors are the columns of P.

(d) $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$. Associated eigenvectors are the columns of P.

12. P is the matrix whose columns are the given eigenvectors:

$$P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}.$$

14. Let A be the given matrix.

(a) Since A is upper triangular its eigenvalues are its diagonal entries. Since $\lambda = 2$ is an eigenvalue of multiplicity 2 we must show, by Theorem 7.4, that it has two linearly independent eigenvectors.

$$(2I_3 - A)\mathbf{x} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row reducing the coefficient we obtain the equivalent linear system

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It follows that there are two arbitrary constants in the general solution so there are two linearly independent eigenvectors. Hence the matrix is diagonalizable.

(b) Since A is upper triangular its eigenvalues are its diagonal entries. Since $\lambda = 2$ is an eigenvalue of multiplicity 2 we must show it has two linearly independent eigenvectors. (We are using Theorem 7.4.)

$$(2I_3 - A)\mathbf{x} = \begin{bmatrix} 0 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row reducing the coefficient matrix we obtain the equivalent linear system

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It follows that there is only one arbitrary constant in the general solution so that there is only one linearly independent eigenvector. Hence the matrix is not diagonalizable.

(c) The matrix is lower triangular hence its eigenvalues are its diagonal entries. Since they are distinct the matrix is diagonalizable.

(d) The eigenvalues of A are $\lambda_1 = 0$ with associated eigenvector $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\lambda_2 = \lambda_3 = 3$, with associated eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Since there are not two linearly independent eigenvectors associated with $\lambda_2 = \lambda_3 = 3$, A is not similar to a diagonal matrix.

16. Each of the given matrices A has a multiple eigenvalue whose associated eigenspace has dimension 1, so the matrix is not diagonalizable.

(a) A is upper triangular with multiple eigenvalue $\lambda_1 = \lambda_2 = 1$ and associated eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) A is upper triangular with multiple eigenvalue $\lambda_1 = \lambda_2 = 2$ and associated eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(c) A has the multiple eigenvalue $\lambda_1 = \lambda_2 = -1$ with associated eigenvector $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

(d) A has the multiple eigenvalue $\lambda_1 = \lambda_2 = 1$ with associated eigenvector $\begin{bmatrix} -3 \\ -7 \\ 8 \\ 0 \end{bmatrix}$.

18. $\begin{bmatrix} 2^9 & 0 \\ 0 & (-2)^9 \end{bmatrix} = \begin{bmatrix} 512 & 0 \\ 0 & -512 \end{bmatrix}$.

20. Necessary and sufficient conditions are: $(a-d)^2 + 4bc > 0$ or that $b = c = 0$ with $a = d$.

Using Theorem 7.4, A is diagonalizable if and only if R^2 has a basis consisting of eigenvectors of A. Thus we must find conditions on the entries of A to guarantee a pair of linearly independent eigenvectors. The characteristic polynomial of A is

$$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

Using the quadratic formula the roots are

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}.$$

Since eigenvalues are required to be real, we require that

$$(a+d)^2 - 4(ad - bc) = a^2 + 2ad + d^2 - 4ad + 4bc = (a-d)^2 + 4bc \geq 0.$$

Suppose first that $(a-d)^2 + 4bc = 0$. Then

$$\lambda = \frac{a+d}{2}$$

is a root of multiplicity 2 and the linear system

$$\left(\frac{a+d}{2} I_2 - A\right)\mathbf{x} = \begin{bmatrix} \dfrac{d-a}{2} & -b \\ -c & \dfrac{a-d}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

must have two linearly independent solutions. A 2×2 homogeneous linear system can have two linearly independent solutions only if the coefficient matrix is the zero matrix. Hence it must follow that $b = c = 0$ and $a = d$. That is, matrix A is a multiple of I_2.

Now suppose $(a-d)^2 + 4bc > 0$. Then the eigenvalues are real and distinct and by Theorem 7.5 A is diagonalizable. Thus, in summary, for A to be diagonalizable it is necessary and sufficient that $(a-d)^2 + 4bc > 0$ or that $b = c = 0$ with $a = d$.

21. Since A and B are nonsingular, A^{-1} and B^{-1} exist. Then $BA = A^{-1}(AB)A$. Therefore AB and BA are similar and hence by Theorem 7.2 they have the same characteristic polynomial. Thus they have the same eigenvalues.

22. The representation of L with respect to the given basis is $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The eigenvalues of L are $\lambda_1 = 1$ and $\lambda_2 = -1$. Associated eigenvectors are e^t and e^{-t}.

23. Let A be diagonalizable with $A = PDP^{-1}$, where D is diagonal.

    (a) $A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = QDQ^{-1}$, where $Q = (P^{-1})^T$. Thus A^T is similar to a diagonal matrix and hence is diagonalizable.

    (b) $A^k = (PDP^{-1})^k = PD^kP^{-1}$. Since D^k is diagonal we have A^k is similar to a diagonal matrix and hence diagonalizable.

24. If A is diagonalizable, then there is a nonsingular matrix P so that $P^{-1}AP = D$, a diagonal matrix. Then $A^{-1} = PD^{-1}P^{-1} = (P^{-1})^{-1}D^{-1}P^{-1}$. Since D^{-1} is a diagonal matrix, we conclude that A^{-1} is diagonalizable.

25. First observe the difference between this result and Theorem 7.5. Theorem 7.5 shows that if *all* the eigenvalues of A are distinct, then the associated eigenvectors are linearly independent. In the present exercise, we are asked to show that if any subset of k eigenvalues are distinct, then the associated eigenvectors are linearly independent. To prove this result, we basically imitate the proof of Theorem 7.5

    Suppose that $S = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is linearly dependent. Then Theorem 4.7 implies that some vector $\mathbf{x}_j$ is a linear combination of the preceding vectors in S. We can assume that $S_1 = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{j-1}\}$ is linearly independent, for otherwise one of the vectors in S_1 is a linear combination of the preceding ones, and we can choose a new set S_2, and so on. We thus have that S_1 is linearly independent and that

$$\mathbf{x}_j = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_{j-1}\mathbf{x}_{j-1}, \tag{1}$$

    where $a_1, a_2, \dots, a_{j-1}$ are real numbers. This means that

$$A\mathbf{x}_j = A(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_{j-1}\mathbf{x}_{j-1}) = a_1A\mathbf{x}_1 + a_2A\mathbf{x}_2 + \cdots + a_{j-1}A\mathbf{x}_{j-1}. \tag{2}$$

    Since $\lambda_1, \lambda_2, \dots, \lambda_j$ are eigenvalues and $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_j$ are associated eigenvectors, we know that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for $i = 1, 2, \dots, n$. Substituting in (2), we have

$$\lambda_j\mathbf{x}_j = a_1\lambda_1\mathbf{x}_1 + a_2\lambda_2\mathbf{x}_2 + \cdots + a_{j-1}\lambda_{j-1}\mathbf{x}_{j-1}. \tag{3}$$

    Multiplying (1) by λ_j, we get

$$\lambda_j\mathbf{x}_j = \lambda_j a_1\mathbf{x}_1 + \lambda_j a_2\mathbf{x}_2 + \cdots + \lambda_j a_{j-1}\mathbf{x}_{j-1}. \tag{4}$$

    Subtracting (4) from (3), we have

$$\mathbf{0} = \lambda_j\mathbf{x}_j - \lambda_j\mathbf{x}_j = a_1(\lambda_1 - \lambda_j)\mathbf{x}_1 + a_2(\lambda_2 - \lambda_j)\mathbf{x}_2 + \cdots + a_{j-1}(\lambda_{j-1} - \lambda_j)\mathbf{x}_{j-1}.$$

    Since S_1 is linearly independent, we must have

$$a_1(\lambda_1 - \lambda_j) = 0, \quad a_2(\lambda_2 - \lambda_j) = 0, \quad \dots, \quad a_{j-1}(\lambda_{j-1} - \lambda_j) = 0.$$

    Now $(\lambda_1 - \lambda_j) \neq 0$, $(\lambda_2 - \lambda_j) \neq 0$, $\dots$, $(\lambda_{j-1} - \lambda_j) \neq 0$, since the λ's are distinct, which implies that

$$a_1 = a_2 = \cdots = a_{j-1} = 0.$$

    This means that $\mathbf{x}_j = \mathbf{0}$, which is impossible if $\mathbf{x}_j$ is an eigenvector. Hence S is linearly independent, so A is diagonalizable.

26. Since B is nonsingular, B^{-1} is nonsingular. It now follows from Exercise 21 that AB^{-1} and $B^{-1}A$ have the same eigenvalues.

27. Let P be a nonsingular matrix such that $P^{-1}AP = D$. Then

$$\mathrm{Tr}(D) = \mathrm{Tr}(P^{-1}AP) = \mathrm{Tr}(P^{-1}(AP)) = \mathrm{Tr}((AP)P^{-1}) = \mathrm{Tr}(APP^{-1}) = \mathrm{Tr}(AI_n) = \mathrm{Tr}(A).$$

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2. (a) A^T. (b) B^T.

3. If $AA^T = I_n$ and $BB^T = I_n$, then
$$(AB)(AB)^T = (AB)(B^TA^T) = A(BB^TA)^T = (AI_n)A^T = AA^T = I_n.$$

4. Since $AA^T = I_n$, then $A^{-1} = A^T$, so $(A^{-1})(A^{-1})^T = (A^{-1})(A^T)^T = (A^{-1})(A) = I_n$.

5. If A is orthogonal then $A^TA = I_n$ so if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are the columns of A, then the (i, j) entry in A^TA is $\mathbf{u}_i^T\mathbf{u}_j$. Thus, $\mathbf{u}_i^T\mathbf{u}_j = 0$ if $i \neq j$ and 1 if $i = j$. Since $\mathbf{u}_i^T\mathbf{u}_j = (\mathbf{u}_i, \mathbf{u}_j)$ then the columns of A form an orthonormal set. Conversely, if the columns of A form an orthonormal set, then $(\mathbf{u}_i, \mathbf{u}_j) = 0$ if $i \neq j$ and 1 if $i = j$. Since $(\mathbf{u}_i, \mathbf{u}_j) = \mathbf{u}_i^T\mathbf{u}_j$, we conclude that $A^TA = I_n$.

6. $AA^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

   $BB^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

7. P is orthogonal since $PP^T = I_3$.

8. If A is orthogonal then $AA^T = I_n$ so $\det(AA^T) = \det(I_n) = 1$ and $\det(A)\det(A^T) = [\det(A)]^2 = 1$, so $\det(A) = \pm 1$.

9. (a) If $A = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$, then $AA^T = I_2$.

   (b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we must have
$$a^2 + b^2 = 1 \qquad (1)$$
$$c^2 + d^2 = 1 \qquad (2)$$
$$ac + bd = 0 \qquad (3)$$
$$ad - bc = \pm 1 \qquad (4)$$
   Let $a = \cos\phi_1$, $b = \sin\phi_1$, $c = \cos\phi_2$, and $d = \sin\phi_2$. Then (1) and (2) hold. From (3) and (4) we obtain
$$\cos(\phi_2 - \phi_1) = 0$$
$$\sin(\phi_2 - \phi_1) = \pm 1.$$
   Thus $\phi_2 - \phi_1 = \pm\dfrac{\pi}{2}$, or $\phi_2 = \phi_1 \pm \dfrac{\pi}{2}$. Then $\cos\phi_2 = \mp\sin\phi_1$ and $\sin\phi_2 = \pm\cos\phi_1$.

10. If $\mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then
$$\begin{aligned}(A\mathbf{x}, A\mathbf{y}) &= \left(\begin{bmatrix} \frac{1}{\sqrt{2}}a_1 & -\frac{1}{\sqrt{2}}a_2 \\ -\frac{1}{\sqrt{2}}a_1 & -\frac{1}{\sqrt{2}}a_2 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}}b_1 & -\frac{1}{\sqrt{2}}b_2 \\ -\frac{1}{\sqrt{2}}b_1 & -\frac{1}{\sqrt{2}}b_2 \end{bmatrix} \right) \\ &= \tfrac{1}{2}(a_1b_1 + a_2b_2 - a_1b_2 - a_2b_1) + \tfrac{1}{2}(a_1b_1 + a_2b_2 + a_1b_2 + a_2b_1) \\ &= a_1b_1 + a_2b_2 \\ &= (\mathbf{x}, \mathbf{y}).\end{aligned}$$

11. We have

$$\cos\theta = \frac{(L(\mathbf{x}), L(\mathbf{y}))}{\|L(\mathbf{x})\|\,\|L(\mathbf{y})\|} = \frac{(A\mathbf{x}, A\mathbf{y})}{\sqrt{(A\mathbf{x}, A\mathbf{x})(A\mathbf{y}, A\mathbf{y})}} = \frac{(\mathbf{x}, \mathbf{y})}{\sqrt{(\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})}} = \frac{(\mathbf{x}, \mathbf{y})}{\|\mathbf{x}\|\,\|\mathbf{y}\|}.$$

12. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$. Recall from Section 5.4 that if S is orthonormal then $(\mathbf{u}, \mathbf{v}) = \left(\left[\mathbf{u}\right]_S, \left[\mathbf{v}\right]_S\right)$, where the latter is the standard inner product on R^n. Now the ith column of A is $\left[L(\mathbf{u}_i)\right]_S$. Then

$$\left(\left[L(\mathbf{u}_i)\right]_S, \left[L(\mathbf{u}_j)\right]_S\right) = (L(\mathbf{u}_i), L(\mathbf{u}_j)) = (\mathbf{u}_i, \mathbf{u}_j) = \left(\left[\mathbf{u}_i\right]_S, \left[\mathbf{u}_j\right]_S\right) = 0$$

if $i \neq j$ and 1 if $i = j$. Hence, A is orthogonal.

13. The representation of L with respect to the natural basis for R^2 is

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

which is orthogonal.

14. If $A\mathbf{x} = \lambda\mathbf{x}$, then $(P^{-1}AP)P^{-1}\mathbf{x} = P^{-1}(\lambda\mathbf{x}) = \lambda(P^{-1}\mathbf{x})$, so that $B(P^{-1}\mathbf{x}) = \lambda(P^{-1}\mathbf{x})$.

16. A is similar to $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

18. A is similar to $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

20. A is similar to $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

22. A is similar to $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

24. A is similar to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

26. A is similar to $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

28. A is similar to $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$.

29. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The characteristic polynomial of A is $p(\lambda) = \lambda^2 - (a+c)\lambda + (ac - b^2)$. The roots of $p(\lambda) = 0$ are

$$\frac{(a+c) + \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} \quad \text{and} \quad \frac{(a+c) - \sqrt{(a+c)^2 - 4(ac-b^2)}}{2}.$$

Case 1. $p(\lambda) = 0$ has distinct real roots and A can then be diagonalized.
Case 2. $p(\lambda) = 0$ has two equal real roots. Then $(a+c)^2 - 4(ac-b^2) = 0$. Since we can write $(a+c)^2 - 4(ac-b^2) = (a-c)^2 + 4b^2$, this expression is zero if and only if $a = c$ and $b = 0$. In this case A is already diagonal.

30. If L is orthogonal, then $\|L(\mathbf{v})\| = \|\mathbf{v}\|$ for any $\mathbf{v}$ in V. If λ is an eigenvalue of L then $L(\mathbf{x}) = \lambda\mathbf{x}$, so $\|L(\mathbf{x})\| = \|\lambda\mathbf{x}\|$, which implies that $\|\lambda\mathbf{x}\| = \|\mathbf{x}\|$. By Exercise 17 of Section 5.3 we then have $|\lambda|\,\|\mathbf{x}\| = \|\mathbf{x}\|$. Since $\mathbf{x}$ is an eigenvector, it cannot be the zero vector, so $|\lambda| = 1$.

31. Let $L\colon R^2 \to R^2$ be defined by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

To show that L is an isometry we verify Equation (7). First note that matrix A satisfies $A^TA = I_2$. (Just perform the multiplication.) Then

$$(L(\mathbf{u}), L(\mathbf{v})) = (A\mathbf{u}, A\mathbf{v}) = (\mathbf{u}, A^TA\mathbf{v}) = (\mathbf{u}, \mathbf{v})$$

so L is an isometry.

32. (a) By Exercise 9(b), if A is an orthogonal matrix and $\det(A) = 1$, then

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

As discussed in Example 8 in Section 1.6, L is then a counterclockwise rotation through the angle ϕ.

(b) If $\det(A) = -1$, then

$$A = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}.$$

Let $L_1\colon R^2 \to R^2$ be reflection about the x-axis. Then with respect to the natural basis for R^2, L_1 is represented by the matrix

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

As we have just seen in part (a), the linear operator L_2 giving a counterclockwise rotation through the angle ϕ is represented with respect to the natural basis for R^2 by the matrix

$$A_2 = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

We have $A = A_2A_1$. Then $L = L_2 \circ L_1$.

33. (a) Let L be an isometry. Then $(L(\mathbf{x}), L(\mathbf{x})) = (\mathbf{x}, \mathbf{x})$, so $\|L(\mathbf{x})\| = \|\mathbf{x}\|$.
    (b) Let L be an isometry. Then the angle θ between $L(\mathbf{x})$ and $L(\mathbf{y})$ is determined by

$$\cos\theta = \frac{(L(\mathbf{x}), L(\mathbf{y}))}{\|L(\mathbf{x})\|\,\|L(\mathbf{y})\|} = \frac{(\mathbf{x}, \mathbf{y})}{\|\mathbf{x}\|\,\|\mathbf{y}\|},$$

    which is the cosine of the angle between $\mathbf{x}$ and $\mathbf{y}$.

34. Let $L(\mathbf{x}) = A\mathbf{x}$. It follows from the discussion preceding Theorem 7.9 that if L is an isometry, then L is nonsingular. Thus, $L^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$. Now

$$(L^{-1}(\mathbf{x}), L^{-1}(\mathbf{y})) = (A^{-1}\mathbf{x}, A^{-1}\mathbf{y}) = (\mathbf{x}, (A^{-1})^T A^{-1}\mathbf{y}).$$

Since A is orthogonal, $A^T = A^{-1}$, so $(A^{-1})^T A^{-1} = I_n$. Thus, $(\mathbf{x}, (A^{-1})^T A^{-1}\mathbf{y}) = (\mathbf{x}, \mathbf{y})$. That is, $(A^{-1}\mathbf{x}, A^{-1}\mathbf{y}) = (\mathbf{x}, \mathbf{y})$, which implies that $(L^{-1}(\mathbf{x}), L^{-1}(\mathbf{y})) = (\mathbf{x}, \mathbf{y})$, so L^{-1} is an isometry.

35. Suppose that L is an isometry. Then $(L(\mathbf{v}_i), L(\mathbf{v}_j)) = (\mathbf{v}_i, \mathbf{v}_j)$, so $(L(\mathbf{v}_i), L(\mathbf{v}_j)) = 1$ if $i = j$ and 0 if $i \neq j$. Hence, $T = \{L(\mathbf{v}_1), L(\mathbf{v}_2), \dots, L(\mathbf{v}_n)\}$ is an orthonormal basis for R^n. Conversely, suppose that T is an orthonormal basis for R^n. Then $(L(\mathbf{v}_i), L(\mathbf{v}_j)) = 1$ if $i = j$ and 0 if $i \neq j$. Thus, $(L(\mathbf{v}_i), L(\mathbf{v}_j)) = (\mathbf{v}_i, \mathbf{v}_j)$, so L is an isometry.

36. Choose $\mathbf{y} = \mathbf{e}_i$, for $i = 1, 2, \dots, n$. Then $A^T A\mathbf{e}_i = \mathrm{Col}_i(A^T A) = \mathbf{e}_i$ for $i = 1, 2, \dots, n$. Hence $A^T A = I_n$.

37. If A is orthogonal, then $A^T = A^{-1}$. Since

$$(A^T)^T = (A^{-1})^T = (A^T)^{-1},$$

we have that A^T is orthogonal.

38. $(cA)^T = (cA)^{-1}$ if and only if $cA^T = \dfrac{1}{c}A^{-1} = \dfrac{1}{c}A^T$. That is, $c = \dfrac{1}{c}$. Hence $c = \pm 1$.

Supplementary Exercises for Chapter 7, p. 477

2. (a) The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -3$, $\lambda_3 = 9$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

   (b) Yes; $P = \begin{bmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{bmatrix}$. P is not unique, since eigenvectors are not unique.
   (c) $\lambda_1 = \frac{1}{3}$, $\lambda_2 = -\frac{1}{3}$, $\lambda_3 = \frac{1}{9}$.
   (d) The eigenvalues are $\lambda_1 = 9$, $\lambda_2 = 9$, $\lambda_3 = 81$. Eigenvectors associated with λ_1 and λ_2 are

$$\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}.$$

   An eigenvector associated with $\lambda_3 = 81$ is $\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$.

3. (a) The characteristic polynomial of A is

$$\det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{12} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n\,n-1} & \lambda - a_{nn} \end{vmatrix}.$$

Any product in $\det(\lambda I_n - A)$, other than the product of the diagonal entries, can contain at most $n-2$ of the diagonal entries of $\lambda I_n - A$. This follows because at least two of the column indices must be out of natural order in every other product appearing in $\det(\lambda I_n - A)$. This implies that the coefficient of λ^{n-1} is formed by the expansion of the product of the diagonal entries. The coefficient of λ^{n-1} is the sum of the coefficients of λ^{n-1} from each of the products

$$-a_{ii}(\lambda - a_{11}) \cdots (\lambda - a_{i-1\,i-1})(\lambda - a_{i+1\,i+1}) \cdots (\lambda - a_{nn})$$

$i = 1, 2, \ldots, n$. The coefficient of λ^{n-1} in each such term is $-a_{ii}$ and so the coefficient of λ^{n-1} in the characteristic polynomial is

$$-a_{11} - a_{22} - \cdots - a_{nn} = -\operatorname{Tr}(A).$$

(b) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then $\lambda - \lambda_i$, $i = 1, 2, \ldots, n$ are factors of the characteristic polynomial $\det(\lambda I_n - A)$. It follows that

$$\det(\lambda I_n - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Proceeding as in (a), the coefficient of λ^{n-1} is the sum of the coefficients of λ^{n-1} from each of the products

$$-\lambda_i(\lambda - \lambda_1) \cdots (\lambda - \lambda_{i-1})(\lambda - \lambda_{i+1}) \cdots (\lambda - \lambda_n)$$

for $i = 1, 2, \ldots, n$. The coefficient of λ^{n-1} in each such term is $-\lambda_i$, so the coefficient of λ^{n-1} in the characteristic polynomial is $-\lambda_1 - \lambda_2 - \cdots - \lambda_n = -\operatorname{Tr}(A)$ by (a). Thus, $\operatorname{Tr}(A)$ is the sum of the eigenvalues of A.

(c) We have

$$\det(\lambda I_n - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

so the constant term is $\pm\lambda_1\lambda_2 \cdots \lambda_n$.

4. $A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1$, but all the eigenvectors are of the form $r \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Clearly A has only one linearly independent eigenvector and is not diagonalizable. However, $\det(A) \neq 0$, so A is nonsingular.

5. In Exercise 21 of Section 7.1 we show that if λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$, then λ^k is an eigenvalue of A^k, k a positive integer. For any positive integers j and k and any scalars a and b, the eigenvalues of $aA^j + bA^k$ are $a\lambda^j + b\lambda^k$. This follows since

$$(aA^j + bA^k)\mathbf{x} = aA^j\mathbf{x} + bA^k\mathbf{x} = a\lambda^j\mathbf{x} + b\lambda^k\mathbf{x} = (a\lambda^j + b\lambda^k)\mathbf{x}.$$

This result generalizes to finite linear combinations of powers of A and to scalar multiples of the identity matrix. Thus,

$$\begin{aligned} p(A)\mathbf{x} &= (a_0 I_n + a_1 A + \cdots + a_k A^k)\mathbf{x} \\ &= a_0 I_n \mathbf{x} + a_1 A\mathbf{x} + \cdots + a_k A^k \mathbf{x} \\ &= a_0 \mathbf{x} + a_1 \lambda \mathbf{x} + \cdots + a_k \lambda^k \mathbf{x} \\ &= (a_0 + a_1 \lambda_1 + \cdots + a_k \lambda^k)\mathbf{x} \\ &= p(\lambda)\mathbf{x}. \end{aligned}$$

6. (a) $p_1(\lambda)p_2(\lambda)$. (b) $p_1(\lambda)p_2(\lambda)$.

8. (a) $\big[L(A_1)\big]_S = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\big[L(A_2)\big]_S = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$, $\big[L(A_3)\big]_S = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $\big[L(A_4)\big]_S = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$.

   (b) $B = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$.

   (c) The eigenvalues of L are $\lambda_1 = -1$, $\lambda_2 = 1$ (of multiplicity 3). An eigenvector associated with $\lambda_1 = -1$ is $\mathbf{x}_1 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$. Eigenvectors associated with $\lambda_2 = 1$ are

$$\mathbf{x}_2 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \quad \text{and} \quad \mathbf{x}_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

   (d) The eigenvalues of L are $\lambda_1 = -1$, $\lambda_2 = 1$ (of multiplicity 3). An eigenvector associated with $\lambda_1 = -1$ is $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$. Eigenvectors associated with $\lambda_2 = 1$ are

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}, \quad \begin{bmatrix}0&1\\1&0\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

   (e) The eigenspace associated with $\lambda_1 = -1$ consists of all matrices of the form

$$\begin{bmatrix}0&k\\-k&0\end{bmatrix},$$

   where k is any real number, that is, it consists of the set of all 2×2 skew symmetric real matrices. The eigenspace associated with $\lambda_2 = 1$ consists of all matrices of the form

$$a\begin{bmatrix}1&0\\0&0\end{bmatrix} + b\begin{bmatrix}0&1\\1&0\end{bmatrix} + c\begin{bmatrix}0&0\\0&1\end{bmatrix},$$

   where a, b, and c are any real numbers, that is, it consists of all 2×2 real symmetric matrices.

10. The eigenvalues of L are $\lambda_1 = 0$, $\lambda_2 = i$, $\lambda_3 = -i$. Associated eigenvectors are $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = i\sin x + \cos x$, $\mathbf{x}_3 = -i\sin x + \cos x$.

11. If A is similar to a diagonal matrix D, then there exists a nonsingular matrix P such that $P^{-1}AP = D$. It follows that

$$D = D^T = (P^{-1}AP)^T = P^TA^T(P^{-1})^T = ((P^T)^{-1})^{-1}A^T(P^T)^{-1},$$

so if we let $Q = (P^T)^{-1}$, then $Q^{-1}A^TQ = D$. Hence, A^T is also similar to D and thus A is similar to A^T.

Chapter Review for Chapter 7, p. 478

True or False

1. True.	2. False.	3. True.	4. True.	5. False.
6. True.	7. True.	8. True.	9. True.	10. True.
11. False.	12. True.	13. True.	14. True.	15. True.
16. True.	17. True.	18. True.	19. True.	20. True.

Quiz

1. $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\lambda_2 = 3$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{5}{4} & \frac{3}{4} \\ 0 & -\frac{3}{4} & -\frac{5}{4} \end{bmatrix}$.

   (b) $\lambda_1 = 0$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\lambda_2 = 1$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$; $\lambda_3 = -1$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$.

3. $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = 2$.

4. $\lambda = 9$, $\mathbf{x}$.

5. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

6. $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

7. No.

8. No.

9. (a) Possible answer: $\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 3 & -1 \\ -1 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$. Thus

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 3 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ -1 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

   Since $\mathbf{z}$ is orthogonal to $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, all entries not on the diagonal of this matrix are zero. The diagonal entries are the squares of the magnitudes of the vectors: $\|\mathbf{x}\|^2 = 6$, $\|\mathbf{y}\|^2 = 18$, and $\|\mathbf{z}\|^2 = 3$.

   (c) Normalize each vector from part (b).

   (d) diagonal

(e) Since

$$A^TA = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{y}^T \\ \mathbf{z}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^T\mathbf{x} & \mathbf{x}^T\mathbf{y} & \mathbf{x}^T\mathbf{z} \\ \mathbf{y}^T\mathbf{x} & \mathbf{y}^T\mathbf{y} & \mathbf{y}^T\mathbf{z} \\ \mathbf{z}^T\mathbf{x} & \mathbf{z}^T\mathbf{y} & \mathbf{z}^T\mathbf{z} \end{bmatrix},$$

it follows that if the columns of A are mutually orthogonal, then all entries of A^TA not on the diagonal are zero. Thus, A^TA is a diagonal matrix.

10. False.

11. Let

$$A = \begin{bmatrix} k & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then $kI_3 - A$ has its first row all zero and hence $\det(kI_3 - A) = 0$. Therefore, $\lambda = k$ is an eigenvalue of A.

12. (a) $\det(4I_3 - A) = \det\left(\begin{bmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{bmatrix}\right) = 0.$

$\det(10I_3 - A) = \det\left(\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}\right) = 0.$

Basis for eigenspace associated with $\lambda = 4$: $\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right\}.$

Basis for eigenspace associated with $\lambda = 10$: $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}.$

(b) $P = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}.$

Chapter 8

Applications of Eigenvalues and Eigenvectors (Optional)

Section 8.1, p. 486

2. $\begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$.

4. (b) and (c)

6. (a) $\mathbf{x}^{(1)} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 0.06 \\ 0.24 \\ 0.70 \end{bmatrix}, \quad \mathbf{x}^{(3)} = \begin{bmatrix} 0.048 \\ 0.282 \\ 0.67 \end{bmatrix}, \quad \mathbf{x}^{(4)} = \begin{bmatrix} 0.0564 \\ 0.2856 \\ 0.658 \end{bmatrix}$.

   (b) $T^3 = \begin{bmatrix} 0.06 & 0.048 & 0.06 \\ 0.3 & 0.282 & 0.282 \\ 0.64 & 0.67 & 0.66 \end{bmatrix}$. Since all entries in T^3 are positive, T is regular. Steady state vector is

$$\mathbf{u} = \begin{bmatrix} \frac{3}{53} \\ \frac{15}{53} \\ \frac{35}{53} \end{bmatrix} \approx \begin{bmatrix} 0.057 \\ 0.283 \\ 0.660 \end{bmatrix}.$$

8. (a) $T^2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$. Since all entries of T^2 are positive, T reaches a state of equilibrium.

   (b) Since all entries of T are positive, it reaches a state of equilibrium.

   (c) $T^2 = \begin{bmatrix} \frac{11}{18} & \frac{1}{3} & \frac{13}{24} \\ \frac{7}{36} & \frac{1}{3} & \frac{11}{48} \\ \frac{7}{36} & \frac{1}{3} & \frac{11}{48} \end{bmatrix}$. Since all entries of T^2 are positive, T reaches a state of equilibrium.

   (d) $T^2 = \begin{bmatrix} 0.2 & 0.05 & 0.1 \\ 0.3 & 0.4 & 0.35 \\ 0.5 & 0.55 & 0.55 \end{bmatrix}$. Since all entries of T^2 are positive, it reaches a state of equilibrium.

10. (a)

$$T = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \end{array}$$

(b) Compute $T\mathbf{x}^{(2)}$, where $\mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$:

$$T\mathbf{x}^{(0)} = \mathbf{x}^{(1)} = \begin{bmatrix} 0.35 \\ 0.65 \end{bmatrix}, \qquad T\mathbf{x}^{(1)} = \mathbf{x}^{(2)} = \begin{bmatrix} 0.365 \\ 0.635 \end{bmatrix}, \qquad T\mathbf{x}^{(2)} = \mathbf{x}^{(3)} = \begin{bmatrix} 0.364 \\ 0.636 \end{bmatrix}.$$

The probability of the rat going through door A on the third day is $p_1^{(3)} = .364$.

(c) $\mathbf{u} = \begin{bmatrix} \frac{4}{11} \\ \frac{7}{11} \end{bmatrix} \approx \begin{bmatrix} 0.364 \\ 0.636 \end{bmatrix}$.

12. red, 25%; pink, 50%; white, 25%.

Section 8.2, p. 500

2. $A = USV^T = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T$

4. $A = USV^T = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$

6. (a) The matrix has rank 3. Its distance from the class of matrices of rank 2 is $s_{\min} = 0.2018$.
   (b) Since $s_{\min} = 0$ and the other two singular values are not zero, the matrix belongs to the class of matrices of rank 2.
   (c) Since $s_{\min} = 0$ and the other three singular values are not zero, the matrix belongs to the class of matrices of rank 3.

7. The singular value decomposition of A is given by $A = USV^T$. From Theorem 8.1 we have

$$\operatorname{rank} A = \operatorname{rank} USV^T = \operatorname{rank} U(SV^T) = \operatorname{rank} SV^T = \operatorname{rank} S.$$

Based on the form of matrix S, its rank is the number of nonzero rows, which is the same as the number of nonzero singular values. Thus $\operatorname{rank} A =$ the number of nonzero singular values of A.

Section 8.3, p. 514

2. (a) The characteristic polynomial was obtained in Exercise 5(d) of Section 7.1: $\lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6)$. So the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 6$. Hence the dominant eigenvalue is 6.
   (b) The eigenvalues were obtained in Exercise 6(d) of Section 7.1: $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 4$. Hence the dominant eigenvalue is 4.

4. (a) 5. (b) 7. (c) 10.

6. (a) $\max\{7, 5\} = 7$. (b) $\max\{7, 4, 5\} = 7$.

7. This is immediate, since $A = A^T$.

8. Possible answer: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

9. We have
$$\|A^k\mathbf{x}\|_1 \le \|A^k\|_1\|\mathbf{x}\|_1 \le \|A\|_1^k\|\mathbf{x}\|_1 \to 0,$$
since $\|A\|_1 < 1$.

10. The eigenvalues of A can all be < 1 in magnitude.

12. Sample mean $= 5825$.
sample variance $= 506875$.
standard deviation $= 711.95$.

14. Sample means $= \begin{bmatrix} 791.8 \\ 826.0 \end{bmatrix}$.
covariance matrix $= \begin{bmatrix} 95{,}996.56 & 76{,}203.00 \\ 76{,}203.00 & 73{,}999.20 \end{bmatrix}$.

16. $S = \begin{bmatrix} 1262200 & 128904 \\ 128904 & 32225.8 \end{bmatrix}$. Eigenvalues and associated eigenvectros:
$$\lambda_1 = 1275560, \quad \mathbf{u}_1 = \begin{bmatrix} 0.9947 \\ 0.1031 \end{bmatrix}$$
$$\lambda_2 = 18861.6; \quad \mathbf{u}_2 = \begin{bmatrix} -0.1031 \\ 0.9947 \end{bmatrix}.$$
$$\text{First principal component} = .9947\text{col}_1(X) + 0.1031\text{col}_2(X) = \begin{bmatrix} 1107.025 \\ 3240.89 \\ 4530.264 \\ 3688.985 \\ 3173.37 \end{bmatrix}.$$

17. Let $\mathbf{x}$ be an eigenvector of C associated with the eigenvalue λ. Then $C\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{x}^TC\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x}$. Hence,
$$\lambda = \frac{\mathbf{x}^TC\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$
We have $\mathbf{x}^TC\mathbf{x} > 0$, since C is positive definite and $\mathbf{x}^T\mathbf{x} > 0$, since $\mathbf{x} \ne \mathbf{0}$. Hence $\lambda > 0$.

18. (a) The diagonal entries of S_n are the sample variances for the n-variables and the total variance is the sum of the sample variances. Since $\text{Tr}(S_n)$ is the sum of the diagonal entries, it follows that $\text{Tr}(S_n) =$ total variance.
(b) S_n is symmetric, so it can be diagonalized by an orthogonal matrix P.
(c) $\text{Tr}(D) = \text{Tr}(P^TS_nP) = \text{Tr}(P^TPS_n) = \text{Tr}(I_nS_n) = \text{Tr}(S_n)$.
(d) Total variance $= \text{Tr}(S_n) = \text{Tr}(D)$, where the diagonal entries of D are the eigenvalues of S_n, so the result follows.

Section 8.4, p. 524

2. (a) $\mathbf{x}(t) = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + b_3 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} e^{3t}.$

(b) $\mathbf{x}(t) = 2\begin{bmatrix}1\\0\\0\end{bmatrix}e^{t} + 3\begin{bmatrix}0\\1\\0\end{bmatrix}e^{-2t} + 4\begin{bmatrix}0\\1\\5\end{bmatrix}e^{3t}.$

4. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to the equation $\mathbf{x}' = A\mathbf{x}$, and let a and b be scalars. Then

$$\frac{d}{dt}(a\mathbf{x}_1 + b\mathbf{x}_2) = a\mathbf{x}_1' + b\mathbf{x}_2' = aA\mathbf{x}_1 + bA\mathbf{x}_2 = A(a\mathbf{x}_1 + b\mathbf{x}_2).$$

Thus $a\mathbf{x}_1 + b\mathbf{x}_2$ is also a solution to the given equation.

6. $\mathbf{x}(t) = b_1\begin{bmatrix}1\\-1\end{bmatrix}e^{5t} + b_2\begin{bmatrix}1\\1\end{bmatrix}e^{t}.$

8. $\mathbf{x}(t) = b_1\begin{bmatrix}0\\-2\\1\end{bmatrix}e^{t} + b_2\begin{bmatrix}1\\0\\0\end{bmatrix}e^{t} + b_3\begin{bmatrix}1\\0\\1\end{bmatrix}e^{3t}.$

10. The system of differential equations is

$$\begin{bmatrix}x'(t)\\y'(t)\end{bmatrix} = \begin{bmatrix}-\frac{1}{10} & \frac{2}{30}\\ \frac{1}{10} & -\frac{2}{30}\end{bmatrix}\begin{bmatrix}x(t)\\y(t)\end{bmatrix}.$$

The characteristic polynomial of the coefficient matrix is $p(\lambda) = \lambda^2 + \frac{1}{6}\lambda$. Eigenvalues and associated eigenvectors are:

$$\lambda_1 = 0, \mathbf{x}_1 = \begin{bmatrix}-1\\1\end{bmatrix}; \quad \lambda_2 = -\tfrac{1}{6}, \mathbf{x}_2 = \begin{bmatrix}\frac{2}{3}\\1\end{bmatrix}.$$

Hence the general solution is given by

$$\begin{bmatrix}x(t)\\y(t)\end{bmatrix} = b_1\begin{bmatrix}-1\\1\end{bmatrix}e^{-\frac{1}{6}t} + b_2\begin{bmatrix}\frac{2}{3}\\1\end{bmatrix}.$$

Using the initial conditions $x(0) = 10$ and $y(0) = 40$, we find that $b_1 = 10$ and $b_2 = 30$. Thus, the particular solution, which gives the amount of salt in each tank at time t, is

$$\begin{aligned}x(t) &= -10e^{-\frac{1}{6}t} + 20\\ y(t) &= 10e^{-\frac{1}{6}t} + 30.\end{aligned}$$

Section 8.5, p. 534

2. The eigenvalues of the coefficient matrix are $\lambda_1 = 2$ and $\lambda_2 = 1$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix}0\\1\end{bmatrix}$. Thus the origin is an unstable equilibrium. The phase portrait shows all trajectories tending away from the origin.

4. The eigenvalues of the coefficient matrix are $\lambda_1 = 1$ and $\lambda_2 = -2$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix}0\\1\end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$. Thus the origin is a saddle point. The phase portrait shows trajectories not in the direction of an eigenvector heading towards the origin, but bending away as $t \to \infty$.

6. The eigenvalues of the coefficient matrix are $\lambda_1 = -1+i$ and $\lambda_2 = -1-i$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ i \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Since the real part of the eigenvalues is negative the origin is a stable equilibrium with trajectories spiraling in towards it.

8. The eigenvalues of the coefficient matrix are $\lambda_1 = -2+i$ and $\lambda_2 = -2-i$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Since the real part of the eigenvalues is negative the origin is a stable equilibrium with trajectories spiraling in towards it.

10. The eigenvalues of the coefficient matrix are $\lambda_1 = 1$ and $\lambda_2 = 5$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus the origin is an unstable equilibrium. The phase portrait shows all trajectories tending away from the origin.

Section 8.6, p. 542

2. (a) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & -3 & 3 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

   (b) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

   (c) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

4. (a) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$. (b) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

6. $y_1^2 + 2y_2^2$.

8. $4y_3^2$.

10. $5y_1^2 - 5y_2^2$.

12. $y_1^2 + y_2^2$.

14. $y_1^2 + y_2^2 + y_3^2$.

16. $y_1^2 + y_2^2 + y_3^2$.

18. $y_1^2 - y_2^2 - y_3^2$; rank $= 3$; signature $= -1$.

20. $y_1^2 = 1$, which represents the two lines $y_1 = 1$ and $y_1 = -1$. The equation $-y_1^2 = 1$ represents no conic at all.

22. g_1, g_2, and g_4 are equivalent. The eigenvalues of the matrices associated with the quadratic forms are: for g_1: 1, 1, -1; for g_2: 9, 3, -1; for g_3: 2, -1, -1; for g_4: 5, 5, -5. The rank r and signature s of g_1, g_2 and g_4 are $r = 3$ and $s = 2p - r = 1$.

24. (d)

25. $(P^TAP)^T = P^TA^TP = P^TAP$ since $A^T = A$.

26. (a) $A = P^T AP$ for $P = I_n$.
    (b) If $B = P^T AP$ with nonsingular P, then $A = (P^{-1})^T BP^{-1}$ and B is congruent to A.
    (c) If $B = P^T AP$ and $C = Q^T BQ$ with P, Q nonsingular, then $C = Q^T P^T APQ = (PQ)^T A(PQ)$ with PQ nonsingular.

27. If A is symmetric, there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal. Since P is orthogonal, $P^{-1} = P^T$. Thus A is congruent to D.

28. Let

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

and let the eigenvalues of A be λ_1 and λ_2. The characteristic polynomial of A is

$$f(\lambda) = \lambda^2 - (a + d)\lambda + ad - b^2.$$

If A is positive definite then both λ_1 and λ_2 are > 0, so $\lambda_1\lambda_2 = \det(A) > 0$. Also,

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a > 0.$$

Conversely, let $\det(A) > 0$ and $a > 0$. Then $\lambda_1\lambda_2 = \det(A) > 0$ so λ_1 and λ_2 are of the same sign. If λ_1 and λ_2 are both < 0 then $\lambda_1 + \lambda_2 = a + d < 0$, so $d < -a$. Since $a > 0$, we have $d < 0$ and $ad < 0$. Now $\det(A) = ad - b^2 > 0$, which means that $ad > b^2$ so $ad > 0$, a contradiction. Hence, λ_1 and λ_2 are both positive.

29. Let A be positive definite and $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. By Theorem 8.10, $g(\mathbf{x})$ is a quadratic form which is equivalent to

$$h(\mathbf{y}) = y_1^2 + y_2^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_r^2.$$

If g and h are equivalent then $h(\mathbf{y}) > 0$ for each $\mathbf{y} \neq \mathbf{0}$. However, this can happen if and only if all terms in $Q'(\mathbf{y})$ are positive; that is, if and only if A is congruent to I_n, or if and only if $A = P^T I_n P = P^T P$.

Section 8.7, p. 551

2. Parabola

4. Two parallel lines.

6. Straight line.

8. Hyperbola.

10. None.

12. Hyperbola; $\dfrac{x'^2}{4} - \dfrac{y'^2}{4} = 1$.

14. Parabola; $x'^2 + 4y' = 0$.

16. Ellipse; $4x'^2 + 5y'^2 = 20$.

18. None; $2x'^2 + y'^2 = -2$.

20. Possible answer: hyperbola; $\dfrac{x'^2}{2} - \dfrac{y'^2}{2} = 1$.

22. Possible answer: parabola; $x'^2 = 4y'$
24. Possible answer: ellipse; $\dfrac{x'^2}{\frac{1}{2}} + y'^2 = 1$.
26. Possible answer: ellipse; $\dfrac{x''^2}{4} + y''^2 = 1$.
28. Possible answer: ellipse; $x''^2 + \dfrac{y''^2}{\frac{1}{2}} = 1$.
30. Possible answer: parabola; $y''^2 = -\frac{1}{8}x''$.

Section 8.8, p. 560

2. Ellipsoid.
4. Elliptic paraboloid.
6. Hyperbolic paraboloid.
8. Hyperboloid of one sheet.
10. Hyperbolic paraboloid.
12. Hyperboloid of one sheet.
14. Ellipsoid.
16. Hyperboloid of one sheet; $\dfrac{x''^2}{8} + \dfrac{y''^2}{4} - \dfrac{z''^2}{8} = 1$.
18. Ellipsoid; $\dfrac{x''^2}{9} + \dfrac{y''^2}{9} + \dfrac{z''^2}{\frac{9}{5}} = 1$.
20. Hyperboloid of two sheets; $x''^2 - y''^2 - z''^2 = 1$.
22. Ellipsoid; $\dfrac{x''^2}{\frac{25}{2}} + \dfrac{y''^2}{\frac{25}{4}} + \dfrac{z''^2}{\frac{25}{10}} = 1$.
24. Hyperbolic paraboloid; $\dfrac{x''^2}{\frac{1}{2}} - \dfrac{y''^2}{\frac{1}{2}} = z''$.
26. Ellipsoid; $\dfrac{x''^2}{\frac{1}{2}} + \dfrac{y''^2}{\frac{1}{2}} + \dfrac{z''^2}{\frac{1}{4}} = 1$.
28. Hyperboloid of one sheet; $\dfrac{x''^2}{4} + \dfrac{y''^2}{2} - \dfrac{z''^2}{1} = 1$.

Chapter 10

MATLAB Exercises

Section 10.1, p. 597

Basic Matrix Properties, p. 598

ML.2. (a) Use command **size(H)**

(b) Just type **H**

(c) Type **H(:,1:3)**

(d) Type **H(4:5,:)**

Matrix Operations, p. 598

ML.2. `aug =`

$$\begin{matrix} 2 & 4 & 6 & -12 \\ 2 & -3 & -4 & 15 \\ 3 & 4 & 5 & -8 \end{matrix}$$

ML.4. (a) **R = A(2,:)**

`R =`

```
3   2   4
```

C = B(:,3)

`C =`

$$\begin{matrix} -1 \\ -3 \\ 5 \end{matrix}$$

V = R ∗ C

`V =`

```
11
```

V is the (2,3)-entry of the product **A ∗ B**.

(b) **C = B(:,2)**

`C =`

```
0
3
2
```

V = A ∗ C

```
V =
     1
    14
     0
    13
```

V is column 2 of the product **A ∗ B**.

(c) **R = A(3,:)**

```
R =
    4   −2    3
```

V = R ∗ B

```
V =
    10   0  17   3
```

V is row 3 of the product **A ∗ B**.

ML.6. (a) Entry-by-entry multiplication.

(b) Entry-by-entry division.

(c) Each entry is squared.

Powers of a Matrix, p. 599

ML.2. (a) **A = tril(ones(5), − 1)**

A

```
ans =
      0   0   0   0   0
      1   0   0   0   0
      1   1   0   0   0
      1   1   1   0   0
      1   1   1   1   0
```

A^2

```
ans =
      0   0   0   0   0
      0   0   0   0   0
      1   0   0   0   0
      2   1   0   0   0
      3   2   1   0   0
```

A^3

```
ans =
      0   0   0   0   0
      0   0   0   0   0
      0   0   0   0   0
      1   0   0   0   0
      3   1   0   0   0
```

A^4

```
ans =
      0   0   0   0   0
      0   0   0   0   0
      0   0   0   0   0
      0   0   0   0   0
      1   0   0   0   0
```

A^5

```
ans =
      0   0   0   0   0
      0   0   0   0   0
      0   0   0   0   0
      0   0   0   0   0
      0   0   0   0   0
```

Thus $k = 5$.

(b) This exercise uses the random number generator **rand**. The matrix A and the value of k may vary.

A = tril(fix(10 ∗ rand(7)),2)

```
A =
    0   0   0   0   0   2   8
    0   0   0   6   7   9   2
    0   0   0   0   3   7   4
    0   0   0   0   0   7   7
    0   0   0   0   0   0   4
    0   0   0   0   0   0   0
    0   0   0   0   0   0   0
```

Here **A^3** is all zeros, so $k = 5$.

ML.4. (a) **(A^2 − 7 ∗ A) ∗ (A + 3 ∗ eye(A))**

```
ans =
      −2.8770   −7.1070   −14.0160
      −4.9360   −5.0480   −14.0160
      −6.9090   −7.1070    −9.9840
```

(b) **(A − eye(A))^2 + (A^3 + A)**

```
ans =
     1.3730   0.2430   0.3840
     0.2640   1.3520   0.3840
     0.1410   0.2430   1.6160
```

(c) Computing the powers of A as $A^2, A^3, \ldots$ soon gives the impression that the sequence is converging to

```
0.2273   0.2727   0.5000
0.2273   0.2727   0.5000
0.2273   0.2727   0.5000
```

Typing **format rat**, and displaying the preceding matrix gives

```
ans =
     5/22   3/11   1/2
     5/22   3/11   1/2
     5/22   3/11   1/2
```

ML.6. The sequence is converging to the zero matrix.

Row Operations and Echelon Forms, p. 600

ML.2. Enter the matrix A into MATLAB and use the following MATLAB commands. We use the **format rat** command to display the matrix A in rational form at each stage.

A = [1/2 1/3 1/4 1/5;1/3 1/4 1/5 1/6;1 1/2 1/3 1/4]

```
A =
    0.5000   0.3333   0.2500   0.2000
    0.3333   0.2500   0.2000   0.1667
    1.0000   0.5000   0.3333   0.2500
```

format rat, A

```
A =
    1/2   1/3   1/4   1/5
    1/3   1/4   1/5   1/6
      1   1/2   1/3   1/4
```

format

(a) **A(1,:) = 2 ∗ A(1,:)**

```
A =
    1.0000   0.6667   0.5000   0.4000
    0.3333   0.2500   0.2000   0.1667
    1.0000   0.5000   0.3333   0.2500
```

format rat, A

```
A =
      1   2/3   1/2   2/5
    1/3   1/4   1/5   1/6
      1   1/2   1/3   1/4
```

format

(b) **A(2,:) = (− 1/3) ∗ A(1,:) + A(2,:)**

```
A =
    1.0000   0.6667   0.5000   0.4000
         0   0.0278   0.0333   0.0333
    1.0000   0.5000   0.3333   0.2500
```

format rat, A

```
A =
    1    2/3    1/2    2/5
    0   1/36   1/30   1/30
    1    1/2    1/3    1/4
```

format

(c) **A(3,:) = −1 ∗ A(1,:) + A(3,:)**

```
A =
    1.0000    0.6667    0.5000    0.4000
         0    0.0278    0.0333    0.0333
         0   -0.1667   -0.1667   -0.1500
```

format rat, A

```
A =
    1    2/3    1/2    2/5
    0   1/36   1/30   1/30
    0   -1/6   -1/6  -3/20
```

format

(d) **temp = A(2,:)**

```
temp =
         0    0.0278    0.0333    0.0333
```

A(2,:) = A(3,:)

```
A =
    1.0000    0.6667    0.5000    0.4000
         0   -0.1667   -0.1667   -0.1500
         0   -0.1667   -0.1667   -0.1500
```

A(3,:) = temp

```
A =
    1.0000    0.6667    0.5000    0.4000
         0   -0.1667   -0.1667   -0.1500
         0    0.0278    0.0333    0.0333
```

format rat, A

```
A =
    1    2/3    1/2    2/5
    0   -1/6   -1/6  -3/20
    0   1/36   1/30   1/30
```

format

ML.4. Enter A into MATLAB, then type **reduce(A)**. Use the menu to select row operations. There are many different sequences of row operations that can be used to obtain the reduced row echelon form. However, the reduced row echelon form is unique and is

```
ans =
    1.0000         0         0    0.0500
         0    1.0000         0   -0.6000
         0         0    1.0000    1.5000
```

format rat, ans

```
ans =
    1    0    0    1/20
    0    1    0    -3/5
    0    0    1     3/2
```

format

ML.6. Enter the augmented matrix **aug** into MATLAB. Then use command **reduce(aug)** to construct row operations to obtain the reduced row echelon form. We obtain

```
ans =
        1  0  1  0  0
        0  1  2  0  0
        0  0  0  0  1
```

The last row is equivalent to the equation $0x + 0y + 0z + 0w = 1$, which is clearly impossible. Thus the system is inconsistent.

ML.8. Enter the augmented matrix **aug** into MATLAB. Then use command **reduce(aug)** to construct row operations to obtain the reduced row echelon form. We obtain

```
ans =
        1  0  −1  0
        0  1   2  0
        0  0   0  0
```

The second row corresponds to the equation $y + 2z = 0$. Hence we can choose z arbitrarily. Set $z = r$, any real real number. Then $y = -2r$. The first row corresponds to the equation $x - z = 0$ which is the same as $x = z = r$. Hence the solution to this system is

$$\begin{aligned} x &= r \\ z &= -2r \\ z &= r \end{aligned}$$

ML.10. After entering A into MATLAB, use command **reduce(− 4 ∗ eye(size(A)) − A)**. Selecting row operations, we can show that the reduced row echelon form of $-4I_2 - A$ is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus the solution to the homogeneous system is

$$\mathbf{x} = \begin{bmatrix} -r \\ r \end{bmatrix}.$$

Hence for any real number r, not zero, we obtain a nontrivial solution.

ML.12. (a) **A = [1 1 1;1 1 0;0 1 1];**
b = [0 3 1]′;
x = A\b

```
x =
     −1
      4
     −3
```

(b) **A = [1 1 1;1 1 − 2;2 1 1];**
b = [1 3 2]′;
x = A\b

```
x =
       1.0000
       0.6667
      −0.0667
```

LU-Factorization, p. 601

ML.2. We show the first few steps of the LU-factorization using routine **lupr** and then display the matrices L and U.

[L,U] = lupr(A)

```
++++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

∗∗∗∗∗ Find an LU-FACTORIZATION by Row Reduction ∗∗∗∗∗

```
L =             U =
   1  0  0         8  −1   2
   0  1  0         3   7   2
   0  0  1         1   1   5
```

OPTIONS

```
<1> Insert element into L.  <-1> Undo previous operation.  <0> Quit.
ENTER your choice ===> 1

Enter multiplier.  -3/8
Enter first row number.  1
Enter number of row that changes.  2
```

```
++++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

Replacement by Linear Combination Complete

```
L =             U =
   1  0  0         8     −1     2
   0  1  0         0  7.375  1.25
   0  0  1         1      1     5
```

You just performed operation $-0.375 * \text{Row}(1) + \text{Row}(2)$

OPTIONS

```
<1> Insert element into L.  <-1> Undo previous operation.  <0> Quit.
ENTER your choice ===> 1
```

```
++++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

Replacement by Linear Combination Complete

```
L =             U =
   1  0  0         8     −1     2
   0  1  0         0  7.375  1.25
   0  0  1         1      1     5
```

You just performed operation $-0.375 * \text{Row}(1) + \text{Row}(2)$

Insert a value in L in the position you just eliminated in U. Let the multiplier you just used be called num. It has the value -0.375.

```
Enter row number of L to change.  2
Enter column number of L to change.  1
Value of L(2,1) = -num
Correct:  L(2,1) = 0.375
```

```
++++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

Continuing the factorization process we obtain

```
L =                        U =
       1       0  0           8     −1      2
   0.375       1  0           0  7.375   1.25
   0.125  0.1525  1           0      0  4.559
```

Warning: It is recommended that the row multipliers be written in terms of the entries of matrix U when entries are decimal expressions. For example, $-U(3,2)/U(2,2)$. This assures that the exact numerical values are used rather than the decimal approximations shown on the screen. The preceding display of L and U appears in the routine **lupr**, but the following displays which are shown upon exit from the routine more accurately show the decimal values in the entries.

```
L =                              U =
    1.0000        0        0          8.0000   −1.0000   2.0000
    0.3750   1.0000        0               0    7.3750   1.2500
    0.1250   0.1525   1.0000               0         0   4.5593
```

ML.4. The detailed steps of the solution of Exercises 7 and 8 are omitted. The solution to Exercise 7 is $\begin{bmatrix}2 & -2 & -1\end{bmatrix}^T$ and the solution to Exercise 8 is $\begin{bmatrix}1 & -2 & 5 & -4\end{bmatrix}^T$.

Matrix Inverses, p. 601

ML.2. We use the fact that A is nonsingular if **rref(A)** is the identity matrix.

(a) **A = [1 2;2 4];**
rref(A)

```
ans =
        1    2
        0    0
```

Thus A is singular.

(b) **A = [1 0 0;0 1 0;1 1 1];**
rref(A)

```
ans =
        1    0    0
        0    1    0
        0    0    1
```

Thus A is nonsingular.

(c) **A = [1 2 1;0 1 2;1 0 0];**
rref(A)

```
ans =
        1    0    0
        0    1    0
        0    0    1
```

Thus A is nonsingular.

ML.4. (a) **A = [2 1;2 3];**
rref(A eye(size(A)))

```
ans =
        1.0000        0     0.7500    −0.2500
             0   1.0000    −0.5000     0.5000
```

format rat, ans

```
ans =
        1    0     3/4    −1/4
        0    1    −1/2     1/2
```

format

(b) **A = [1 − 1 2;0 2 1;1 0 0];**
rref(A eye(size(A)))

```
ans =
        1.0000        0        0         0        0    1.0000
             0   1.0000        0   −0.2000   0.4000    0.2000
             0        0   1.0000    0.4000   0.2000   −0.4000
```

format rat, ans

```
ans  =
        1    0    0        0      0       1
        0    1    0     −1/5    2/5     1/5
        0    0    1      2/5    1/5    −2/5
```

format

Determinants by Row Reduction, p. 601

ML.2. There are many sequences of row operations that can be used. Here we record the value of the determinant so you may check your result.

(a) $\det(A) = -9$. (b) $\det(A) = 5$.

ML.4. (a) **A = [2 3 0;4 1 0;0 0 5];**

det(5 ∗ eye(size(A)) − A)

```
ans =
      0
```

(b) **A = [1 1;5 2];**

det(3 ∗ eye(size(A)) − A)^2

```
ans =
      9
```

(c) **A = [1 1 0;0 1 0;1 0 1];**

det(inverse(A) ∗ A)

```
ans =
      1
```

Determinants by Cofactor Expansion, p. 602

ML.2. **A = [1 5 0;2 −1 3;3 2 1];**

cofactor(2,1,A)	**cofactor(2,2,A)**	**cofactor(2,3,A)**
ans =	ans =	ans =
−5	1	13

ML.4. **A = [−1 2 0 0;2 −1 2 0; 0 2 −1 2;0 0 2 −1];**

(Use expansion about the first column.)

detA = −1 ∗ cofactor(1,1,A) + 2 ∗ cofactor(2,1,A)

```
detA  =
       5
```

Vector Spaces, p. 603

ML.2. **p = [2 5 1 −2],q = [1 0 3 5]**

```
p  =
       2    5    1   −2

q  =
       1    0    3    5
```

(a) **p + q**

```
ans  =
        3    5    4    3
```

which is $3t^3 + 5t^2 + 4t + 3$.

(b) **5 ∗ p**

```
ans =
          10    25    5    −10
```

which is $10t^3 + 25t^2 + 5t - 10$.

(c) **3 ∗ p − 4 ∗ q**

```
ans =
          2    15    −9    −26
```

which is $2t^3 + 15t^2 - 9t - 26$.

Subspaces, p. 603

ML.4. (a) Apply the procedure in ML.3(a).

v1 = [1 2 1];v2 = [3 0 1];v3 = [1 8 3];v = [−2 14 4];
rref([v1′ v2′ v3′ v′])

```
ans =
     1   0    4    7
     0   1   −1   −3
     0   0    0    0
```

This system is consistent so $\mathbf{v}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In the general solution if we set $c_3 = 0$, then $c_1 = 7$ and $c_2 = 3$. Hence $7\mathbf{v}_1 - 3\mathbf{v}_2 = \mathbf{v}$. There are many other linear combinations that work.

(b) After entering the 2×2 matrices into MATLAB we associate a column with each one by 'reshaping' it into a 4×1 matrix. The linear system obtained from the linear combination of reshaped vectors is the same as that obtained using the 2×2 matrices in $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$.

v1 = [1 2;1 0];v2 = [2 −1;1 2];v3 = [−3 1;0 1];v = eye(2);
rref([reshape(v1,4,1) reshape(v2,4,1) reshape(v3,4,1) reshape(v,4,1)])

```
ans =
     1   0   0   0
     0   1   0   0
     0   0   1   0
     0   0   0   1
```

The system is inconsistent, hence $\mathbf{v}$ is not a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

ML.6. Follow the method in ML.4(a).

v1 = [1 1 0 1]; v2 = [1 −1 0 1]; v3 = [0 1 2 1];

(a) **v = [2 3 2 3];**
rref([v1′ v2′ v3′ v′])

```
ans =
     1   0   0   2
     0   1   0   0
     0   0   1   1
     0   0   0   0
```

Since the system is consistent, $\mathbf{v}$ is in span S. In fact, $\mathbf{v} = 2\mathbf{v}_1 + \mathbf{v}_3$.

(b) **v = [2 −3 −2 3];**
rref([v1′ v2′ v3′ v′])

```
ans =
      1  0  0  0
      0  1  0  0
      0  0  1  0
      0  0  0  1
```

The system is inconsistent, hence **v** is not in span S.

(c) **v = [0 1 2 3];**
rref([v1′ v2′ v3′ v′])

```
ans =
      1  0  0  0
      0  1  0  0
      0  0  1  0
      0  0  0  1
```

The system is inconsistent, hence **v** is not in span S.

Linear Independence/Dependence, p. 604

ML.2. Form the augmented matrix $\left[A \mid \mathbf{0}\right]$ and row reduce it.

A = [1 2 0 1;1 1 1 2;2 −1 5 7;0 2 −2 −2];

rref([A zeros(4,1)])

```
ans =
      1  0  2  3  0
      0  1 -1 -1  0
      0  0  0  0  0
      0  0  0  0  0
```

The general solution is $x_4 = s$, $x_3 = t$, $x_2 = t + s$, $x_1 = -2t - 3s$. Hence

$$\mathbf{x} = \begin{bmatrix} -2t-3s & t+s & t & s \end{bmatrix}' = t\begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix}' + s\begin{bmatrix} -3 & 1 & 0 & 1 \end{bmatrix}'$$

and it follows that $\begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix}'$ and $\begin{bmatrix} -3 & 1 & 0 & 1 \end{bmatrix}'$ span the solution space.

Bases and Dimension, p. 604

ML.2. Follow the procedure in Exercise ML.5(b) in Section 5.2.

v1 = [0 2 −2]′;v2 = [1 −3 1]′;v3 = [2 −8 4]′;

rref([v1 v2 v3 zeros(size(v1))])

```
ans =
      1  0  -1  0
      0  1   2  0
      0  0   0  0
```

It follows that there is a nontrivial solution so S is linearly dependent and cannot be a basis for V.

ML.4. Here we do not know dim(span S), but dim(span S) = the number of linearly independent vectors in S. We proceed as we did in ML.1.

v1 = [1 2 1 0]′;v2 = [2 1 3 1]′;v3 = [2 −2 4 2]′;

rref([v1 v2 v3 zeros(size(v1))])

```
ans =
        1   0   −2   0
        0   1    2   0
        0   0    0   0
        0   0    0   0
```

The leading 1's imply that $\mathbf{v}_1$ and $\mathbf{v}_2$ are a linearly independent subset of S, hence dim(span S) = 2 and S is not a basis for V.

ML.6. Any vector in V has the form

$$\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a & 2a-c & c \end{bmatrix} = a\begin{bmatrix} 1 & 2 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}.$$

It follows that $T = \left\{\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}\right\}$ spans V and since the members of T are not multiples of one another, T is a linearly independent subset of V. Thus $\dim V = 2$. We need only determine if S is a linearly independent subset of V. Let

v1 = [0 1 − 1]′;v2 = [1 1 1]′;

then

rref([v1 v2 zeros(size(v1))])

```
ans =
        1   0   0
        0   1   0
        0   0   0
```

It follows that S is linearly independent and so Theorem 4.9 implies that S is a basis for V.

In Exercises ML.7 through ML.9 we use the technique involving leading 1's as in Example 5.

ML.8. Associate a column with each 2×2 matrix as in Exercise ML.4(b) in Section 5.2.

v1 = [1 2;1 2]′;v2 = [1 0;1 1]′;v3 = [0 2;0 1]′;v4 = [2 4;2 4]′;v5 = [1 0;0 1]′;

rref([reshape(v1,4,1) reshape(v2,4,1) reshape(v3,4,1) reshape(v4,4,1) reshape(v5,4,1) zeros(4,1)])

```
ans =
        1   0    1   2   0   0
        0   1   −1   0   0   0
        0   0    0   0   1   0
        0   0    0   0   0   0
```

The leading 1's point to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_5$ which are a basis for span S. We have dim(span S) = 3 and span $S \neq M_{22}$.

ML.10. **v1 = [1 1 0 0]′;v2 = [1 0 1 0]′;**

rref([v1 v2 eye(4) zeros(size(v1))])

```
ans =
        1   0   0    1    0   0   0
        0   1   0    0    1   0   0
        0   0   1   −1   −1   0   0
        0   0   0    0    0   1   0
```

It follows that $\left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}', \mathbf{e}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}'\right\}$ is a basis for V which contains S.

ML.12. Any vector in V has the form $\begin{bmatrix} a & 2d+e & a & d & e \end{bmatrix}$. It follows that

$$\begin{bmatrix} a & 2d+e & a & d & e \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and $T = \left\{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix}\right\}$ is a basis for V. Hence let

v1 = [0 3 0 2 − 1]′;w1 = [1 0 1 0 0]′;w2 = [0 2 0 1 0]′;w3 = [0 1 0 0 1]′;

then

rref([v1 w1 w2 w3 eye(4) zeros(size(v1))])

```
ans =
      1    0    0 −1    0
      0    1    0    0    0
      0    0    1    2    0
      0    0    0    0    0
      0    0    0    0    0
```

Thus $\{\mathbf{v}_1, \mathbf{w}_1, \mathbf{w}_2\}$ is a basis for V containing S.

Coordinates and Change of Basis, p. 605

ML.2. Proceed as in ML.1 by making each of the vectors in S a column in matrix A.

A = [1 0 1 1;1 2 1 3;0 2 1 1;0 1 0 0]′;

rref(A)

```
ans =
      1    0    0    0
      0    1    0    0
      0    0    1    0
      0    0    0    1
```

To find the coordinates of $\mathbf{v}$ we solve a linear system. We can do all three parts simultaneously as follows. Associate with each vector $\mathbf{v}$ a column. Form a matrix B from these columns.

B = [4 12 8 14;1/2 0 0 0;1 1 1 7/3]′;

rref([A B])

```
ans =
      1.0000         0         0         0    1.0000    0.5000    0.3333
           0    1.0000         0         0    3.0000         0    0.6667
           0         0    1.0000         0    4.0000   −0.5000         0
           0         0         0    1.0000   −2.0000    1.0000   −0.3333
```

The coordinates are the last three columns of the preceding matrix.

ML.4. **A = [1 0 1;1 1 0;0 1 1];**

B = [2 1 1;1 2 1;1 1 2];

rref([A B])

```
ans =
      1    0    0    1    1    0
      0    1    0    0    1    1
      0    0    1    1    0    1
```

The transition matrix from the T-basis to the S-basis is $P =$ **ans(:,4:6)**.

```
P =
    1   1   0
    0   1   1
    1   0   1
```

ML.6. **A = [1 2 3 0;0 1 2 3;3 0 1 2;2 3 0 1]′;**
B = eye(4);
rref([A B])

```
ans =
   1.0000        0        0        0   0.0417   0.0417   0.2917  −0.2083
        0   1.0000        0        0  −0.2083   0.0417   0.0417   0.2917
        0        0   1.0000        0   0.2917  −0.2083   0.0417   0.0417
        0        0        0   1.0000   0.0417   0.2917  −0.2083   0.0417
```

The transition matrix P is found in columns 5 through 8 of the preceding matrix.

Homogeneous Linear Systems, p. 606

ML.2. Enter A into MATLAB and we find that
rref(A)

```
ans =
     1   0   0
     0   1   0
     0   0   1
     0   0   0
     0   0   0
```

The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

ML.4. Form the matrix $3I_2 - A$ in MATLAB as follows.
C = 3 ∗ eye(2) − [1 2;2 1]

```
C =
     2  −2
    −2   2
```

rref(C)

```
ans =
     1  −1
     0   0
```

The solution is $\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix}$, for t any real number. Just choose $t \neq 0$ to obtain a nontrivial solution.

Rank of a Matrix, p. 606

ML.2. (a) One basis for the row space of A consists of the nonzero rows of rref(A).
A = [1 3 1;2 5 0;4 11 2;6 9 1];
rref(A)

```
ans =
        1   0   0
        0   1   0
        0   0   1
        0   0   0
```

Another basis is found using the leading 1's of rref(A^T) to point to rows of A that form a basis for the row space of A.

rref(A′)

```
ans =
        1   0   2   0
        0   1   1   0
        0   0   0   1
```

It follows that rows 1, 2, and 4 of A are a basis for the row space of A.

(b) Follow the same procedure as in part (a).

A = [2 1 2 0;0 0 0 0;1 2 2 1;4 5 6 2;3 3 4 1];

```
ans =
        1.0000        0   0.6667   −0.3333
             0   1.0000   0.6667    0.6667
             0        0        0         0
             0        0        0         0
```

format rat, ans

```
ans =
        1   0   2/3   −1/3
        0   1   2/3    2/3
        0   0     0      0
        0   0     0      0
```

format

rref(A′)

```
ans =
        1   0   0   1   1
        0   0   1   2   1
        0   0   0   0   0
        0   0   0   0   0
```

It follows that rows 1 and 2 of A are a basis for the row space of A.

ML.4. (a) **A = [3 2 1;1 2 − 1;2 1 3];**

rank(A)

```
ans =
        3
```

The nullity of A is 0.

(b) **A = [1 2 1 2 1;2 1 0 0 2;1 − 1 − 1 − 2 1;3 0 − 1 − 2 3];**

rank(A)

```
ans =
        2
```

The nullity of $A = 5 - \text{rank}(A) = 3$.

Standard Inner Product, p. 607

ML.2. (a) $\mathbf{u = [2 \;\; 2 \;\; -1]';norm(u)}$
```
ans =
     3
```

(b) $\mathbf{v = [0 \;\; 4 \;\; -3 \;\; 0]';norm(v)}$
```
ans =
     5
```

(c) $\mathbf{w = [1 \;\; 0 \;\; 1 \;\; 0 \;\; 3]';norm(w)}$
```
ans =
     3.3166
```

ML.4. Enter A, B, and C as points and construct vectors $\mathbf{v}AB$, $\mathbf{v}BC$, and $\mathbf{v}CA$. Then determine the lengths of the vectors.

$\mathbf{A = [1 \;\; 3 \;\; -2];B = [4 \;\; -1 \;\; 0];C = [1 \;\; 1 \;\; 2];}$
$\mathbf{vAB = B - C}$
```
vAB =
     3    −2    −2
```
$\mathbf{norm(vAB)}$
```
ans =
     4.1231
```

$\mathbf{vBC = C - B}$
```
vBC =
     −3    2    2
```
$\mathbf{norm(vBC)}$
```
ans =
     4.1231
```

$\mathbf{vCA = A - C}$
```
vCA =
     0    2    −4
```
$\mathbf{norm(vCA)}$
```
ans =
     4.4721
```

ML.8. (a) $\mathbf{u = [3 \;\; 2 \;\; 4 \;\; 0];v = [0 \;\; 2 \;\; -1 \;\; 0];}$
$\mathbf{ang = dot(u,v)/((norm(u) * norm(v))}$
```
ang =
     0
```

(b) $\mathbf{u = [2 \;\; 2 \;\; -1];v = [2 \;\; 0 \;\; 1];}$
$\mathbf{ang = dot(u,v)/((norm(u) * norm(v))}$
```
ang =
     0.4472
```
$\mathbf{degrees = ang * (180/pi)}$
```
degrees =
     25.6235
```

(c) $\mathbf{u = [1 \;\; 0 \;\; 0 \;\; 2];v = [0 \;\; 3 \;\; -4 \;\; 0];}$
$\mathbf{ang = dot(u,v)/((norm(u) * norm(v))}$
```
ang =
     0
```

Cross Product, p. 608

ML.2. (a) $\mathbf{u = [2 \;\; 3 \;\; -1];v = [2 \;\; 3 \;\; 1];cross(u,v)}$

```
ans =
        6   -4   0
```

(b) $\mathbf{u = [3 \;\; -1 \;\; 1];v = 2*u;cross(u,v)}$

```
ans =
        0   0   0
```

(c) $\mathbf{u = [1 \;\; -2 \;\; 1];v = [3 \;\; 1 \;\; -1];cross(u,v)}$

```
ans =
        1   4   7
```

ML.4. Following Example 6 we proceed as follows in MATLAB.

$\mathbf{u = [3 \;\; -2 \;\; 1];v = [1 \;\; 2 \;\; 3];w = [2 \;\; -1 \;\; 2];}$
vol = abs(dot(u,cross(v,w)))

```
vol =
       8
```

The Gram-Schmidt Process, p. 608

ML.2. Use the following MATLAB commands.

$\mathbf{A = [1 \;\; 0 \;\; 1 \;\; 1;1 \;\; 2 \;\; 1 \;\; 3;0 \;\; 2 \;\; 1 \;\; 1;0 \;\; 1 \;\; 0 \;\; 0]';}$

gschmidt(A)

```
ans =
        0.5774   -0.2582   -0.1690    0.7559
             0    0.7746    0.5071    0.3780
        0.5774   -0.2582    0.6761   -0.3780
        0.5774    0.5164   -0.5071   -0.3780
```

ML.4. We have that all vectors of the form $\begin{bmatrix} a & 0 & a+b & b+c \end{bmatrix}$ can be expressed as follows:

$$\begin{bmatrix} a & 0 & a+b & b+c \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

By the same type of argument used in Exercises 16–19 we show that

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\right\}$$

is a basis for the subspace. Apply routine **gschmidt** to the vectors of S.

$\mathbf{A = [1 \;\; 0 \;\; 1 \;\; 0;0 \;\; 0 \;\; 1 \;\; 1;0 \;\; 0 \;\; 0 \;\; 1]';}$

gschmidt(A,1)

```
ans =
        1.0000   -0.5000    0.3333
             0         0         0
        1.0000    0.5000   -0.3333
             0    1.0000    0.3333
```

The columns are an orthogonal basis for the subspace.

Projections, p. 609

ML.2. **w1 = [1 0 1 1]′,w2 = [1 1 − 1 0]′**

```
w1 =
     1
     0
     1
     1

w2 =
      1
      1
     -1
      0
```

(a) We show the dot product of w1 and w2 is zero and since nonzero orthogonal vectors are linearly independent they form a basis for W.

dot(w1,w2)

```
ans =
     0
```

(b) **v = [2 1 2 1]′**

```
v =
     2
     1
     2
     1
```

proj = dot(v,w1)/norm(w1)^2 ∗ w1

```
proj =
     1.6667
          0
     1.6667
     1.6667
```

format rat

proj

```
proj =
     5/3
       0
     5/3
     5/3
```

format

(c) **proj = dot(v,w1)/norm(w1)^2 ∗ w1 + dot(v,w2)/norm(w2)^2 ∗ w2**

```
proj =
     2.0000
     0.3333
     1.3333
     1.6667
```

format rat

proj

```
proj =
      2
    1/3
    4/3
    5/3
```

format

ML.4. Note that the vectors in S are not an orthogonal basis for $W = \text{span } S$. We first use the Gram–Schmidt process to find an orthonormal basis.

x = [[1 1 0 1]′ [2 −1 0 0]′ [0 1 0 1]′]

```
x =
    1    2   0
    1   −1   1
    0    0   0
    1    0   1
```

b = gschmidt(x)

```
x =
    0.5774    0.7715   −0.2673
    0.5774   −0.6172   −0.5345
         0         0         0
    0.5774   −0.1543    0.8018
```

Name these columns w1, w2, w3, respectively.

w1 = b(:,1);w2 = b(:,2);w3 = b(:,3);

Then w1, w2, w3 is an orthonormal basis for W.

v = [0 0 1 1]′

```
v =
    0
    0
    1
    1
```

(a) **proj = dot(v,w1) ∗ w1 + dot(v,w2) ∗ w2 + dot(v,w3) ∗ w3**

```
proj =
    0.0000
         0
         0
    1.0000
```

(b) The distance from **v** to P is the length of vector $-\text{proj} + \mathbf{v}$.

norm(− proj + v)

```
ans =
    1
```

Least Squares, p. 609

ML.2. (a) $y = 331.44x + 18704.83$. (b) 24007.58.

ML.4. Data for quadratic least squares: (Sample of cos on $[0, 1.5 * \text{pi}]$.)

```
t          yy
0           1.0000
0.5000      0.8800
1.0000      0.5400
1.5000      0.0700
2.0000     −0.4200
2.5000     −0.8000
3.0000     −0.9900
3.5000     −0.9400
4.0000     −0.6500
4.5000     −0.2100
```

v = polyfit(t,yy,2)

```
v =
    0.2006   −1.2974   1.3378
```

Thus $y = 0.2006t^2 - 1.2974t + 1.3378$.

Kernel and Range of Linear Transformations, p. 611

ML.2. **A = [− 3 2 − 7;2 − 1 4;2 − 2 6];**

rref(A)

```
ans =
      1   0    1
      0   1   −2
      0   0    0
```

It follows that the general solution to $A\mathbf{x} = \mathbf{0}$ is obtained from

$$\begin{aligned} x_1 \quad + \ x_3 &= 0 \\ x_2 - 2x_3 &= 0. \end{aligned}$$

Let $x_3 = r$, then $x_2 = 2r$ and $x_1 = -r$. Thus

$$\mathbf{x} = \begin{bmatrix} -r \\ 2r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

and $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is a basis for $\ker L$. To find a basis for range L proceed as follows.

rref(A′)′

```
ans =
      1    0   0
      0    1   0
     −2   −2   0
```

Then $\left\{\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right\}$ is a basis for range L.

Matrix of a Linear Transformation, p. 611

ML.2. Enter C and the vectors from the S and T bases into MATLAB. Then compute the images of $\mathbf{v}_i$ as $L(\mathbf{v}_i) = C * \mathbf{v}_i$.

C = [1 2 0;2 1 − 1;3 1 0; − 1 0 2]

```
C =
     1   2    0
     2   1   −1
     3   1    0
    −1   0    2
```

v1 = [1 0 1]′; v2 = [2 0 1]′; v3 = [0 1 2]′;

w1 = [1 1 1 2]′; w2 = [1 1 1 0]′; w3 = [0 1 1 − 1]′; w4 = [0 0 1 0]′;

Lv1 = C ∗ v1; Lv2 = C ∗ v2; Lv3 = C ∗ v3;

rref([w1 w2 w3 w4 Lv1 Lv2 Lv3])

```
ans =
    1.0000        0        0        0   0.5000   0.5000    0.5000
         0   1.0000        0        0   0.5000   1.5000    1.5000
         0        0   1.0000        0        0   1.0000   −3.0000
         0        0        0   1.0000   2.0000   3.0000    2.0000
```

It follows that A consists of the last 3 columns of ans.

A = ans(:,5:7)

```
A =
   0.5000   0.5000    0.5000
   0.5000   1.5000    1.5000
        0   1.0000   −3.0000
   2.0000   3.0000    2.0000
```

Eigenvalues and Eigenvectors, p. 612

ML.2. The eigenvalues of matrix A will be computed using MATLAB command **roots(poly(A))**.

(a) **A = [1 − 3;3 − 5];**
r = roots(poly(A))

```
r =
    −2
    −2
```

(b) **A = [3 − 1 4; − 1 0 1;4 1 2];**
r = roots(poly(A))

```
r =
     6.5324
    −2.3715
     0.8392
```

(c) $\mathbf{A = [2 \quad -2 \quad 0;1 \quad -1 \quad 0;1 \quad -1 \quad 0];}$
$\mathbf{r = roots(poly(A))}$

```
r =
   0
   0
   1
```

(d) $\mathbf{A = [2 \quad 4;3 \quad 6];}$
$\mathbf{r = roots(poly(A))}$

```
r =
   0
   8
```

ML.4. (a) $\mathbf{A = [0 \quad 2;-1 \quad 3];}$
$\mathbf{r = roots(poly(A))}$

```
r =
   2
   1
```

The eigenvalues are distinct, so A is diagonalizable. We find the corresponding eigenvectors.
$\mathbf{M = (\ 2 * eye(size(A)) - A)}$
$\mathbf{rref([M\ [0 \quad 0]'])}$

```
ans =
      1   -1   0
      0    0   0
```

The general solution is $x_2 = r$, $x_1 = x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix} 1 & 1 \end{bmatrix}'$ is an eigenvector.
$\mathbf{M = (1 * eye(size(A)) - A)}$
$\mathbf{rref([M\ [0 \quad 0]'])}$

```
ans =
      1   -2   0
      0    0   0
```

The general solution is $x_2 = r$, $x_1 = 2x_2 = 2r$. Let $r = 1$ and we have that $\begin{bmatrix} 2 & 1 \end{bmatrix}'$ is an eigenvector.
$\mathbf{P = [1 \quad 1;2 \quad 1]'}$

```
P =
   1   2
   1   1
```

$\mathbf{invert(P) * A * P}$

```
ans =
      2   0
      0   1
```

(b) $\mathbf{A = [1 \quad -3;3 \quad -5];}$
$\mathbf{r = roots(poly(A))}$

```
r =
   -2
   -2
```

M = (− 2 ∗ eye(size(A)) − A)
rref([M [0 0]′])

ans =

1	−1	0
0	0	0

The general solution is $x_2 = r$, $x_1 = x_2 = r$. Let $r = 1$ and it follows that $\begin{bmatrix}1 & 1\end{bmatrix}'$ is an eigenvector, but there is only one linearly independent eigenvector. Hence A is not diagonalizable.

(c) **A = [0 0 4;5 3 6;6 0 5];**
r = roots(poly(A))

r =

8.0000
3.0000
−3.0000

The eigenvalues are distinct, thus A is diagonalizable. We find the corresponding eigenvectors.

M = (8 ∗ eye(size(A)) − A)
rref([M [0 0 0]′])

ans =

1.0000	0	−0.5000	0
0	1.0000	−1.7000	0
0	0	0	0

The general solution is $x_3 = r$, $x_2 = 1.7x_3 = 1.7r$, $x_1 = .5x_3 = .5r$. Let $r = 1$ and we have that $\begin{bmatrix}.5 & 1.7 & 1\end{bmatrix}'$ is an eigenvector.

M = (3 ∗ eye(size(A)) − A)
rref([M [0 0 0]′])

ans =

1	0	0	0
0	0	1	0
0	0	0	0

Thus $\begin{bmatrix}0 & 1 & 0\end{bmatrix}'$ is an eigenvector.

M = (− 3 ∗ eye(size(A)) − A)
rref([M [0 0 0]′])

ans =

1.0000	0	1.3333	0
0	1.0000	−0.1111	0
0	0	0	0

The general solution is $x_3 = r$, $x_2 = \frac{1}{9}x_3 = \frac{1}{9}r$, $x_1 = -\frac{4}{3}x_3 = -\frac{4}{3}r$. Let $r = 1$ and we have that $\begin{bmatrix}-\frac{4}{3} & \frac{1}{9} & 1\end{bmatrix}'$ is an eigenvector. Thus P is

P = [.5 1.7 1;0 1 0; − 4/3 1/9 1]′
invert(P) ∗ A ∗ P

ans =

8	0	0
0	3	0
0	0	−3

ML.6. $\mathbf{A = [-1 \;\; 1.5 \;\;\; -1.5; -2 \;\; 2.5 \;\;\; -1.5; -2 \;\; 2.0 \;\;\; -1.0]'}$

$\mathbf{r = roots(poly(A))}$

```
r =
      1.0000
     −1.0000
      0.5000
```

The eigenvalues are distinct, hence A is diagonalizable.

$\mathbf{M = (1 * eye(size(A)) - A)}$

$\mathbf{rref([M \;\; [0 \;\; 0 \;\; 0]'0)}$

```
ans =
       1   0    0   0
       0   1   −1   0
       0   0    0   0
```

The general solution is $x_3 = r$, $x_2 = r$, $x_1 = 0$. Let $r = 1$ and we have that $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}'$ is an eigenvector.

$\mathbf{M = (-1 * eye(size(A)) - A)}$

$\mathbf{rref([M \;\; [0 \;\; 0 \;\; 0]')}$

```
ans =
       1   0   −1   0
       0   1   −1   0
       0   0    0   0
```

The general solution is $x_3 = r$, $x_2 = r$, $x_1 = r$. Let $r = 1$ and we have that $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$ is an eigenvector.

$\mathbf{M = (.5 * eye(size(A)) - A)}$

$\mathbf{rref([M \;\; [0 \;\; 0 \;\; 0]')}$

```
ans =
       1   −1   0   0
       0    0   1   0
       0    0   0   0
```

The general solution is $x_3 = 0$, $x_2 = r$, $x_1 = r$. Let $r = 1$ and we have that $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}'$ is an eigenvector. Hence let

$\mathbf{P = [0 \;\; 1 \;\; 1; 1 \;\; 1 \;\; 1; 1 \;\; 1 \;\; 0]'}$

```
P =
    0   1   1
    1   1   1
    1   1   0
```

then we have

$\mathbf{A30 = P * (diag([1 - 1 \;\; .5])\hat{\ }30 * invert(P))}$

```
A30 =
       1.0000   −1.0000   1.0000
            0    0.0000   1.0000
            0         0   1.0000
```

Since all the entries are not displayed as integers we set the format to long and redisplay the matrix to view its contents for more detail.

format long

A30

```
A30 =
   1.0000000000000  -0.99999999906868   0.99999999906868
                 0   0.00000000093132   0.99999999906868
                 0                  0   1.00000000000000
```

Note that this is not the same as the matrix A30 in Exercise ML.5.

Diagonalization, p. 613

ML.2. (a) **A = [1 2; − 1 4];**

[V,D] = eig(A)

```
V =
   -0.8944   -0.7071
   -0.4472   -0.7071

D =
   2   0
   0   3
```

V′ ∗ V

```
ans =
   1.0000   0.9487
   0.9487   1.0000
```

Hence V is not orthogonal. However, since the eigenvalues are distinct A is diagonalizable, so V can be replaced by an orthogonal matrix.

(b) **A = [2 1 2;2 2 − 2;3 1 1];**

[V,D] = eig(A)

```
V =
   -0.5482    0.7071    0.4082
    0.6852   -0.0000   -0.8165
    0.4796    0.7071    0.4082

D =
   -1.0000         0         0
         0    4.0000         0
         0         0    2.0000
```

V′ ∗ V

```
ans =
    1.0000   -0.0485   -0.5874
   -0.0485    1.0000    0.5774
   -0.5874    0.5774    1.0000
```

Hence V is not orthogonal. However, since the eigenvalues are distinct A is diagonalizable, so V can be replaced by an orthogonal matrix.

(c) **A = [1 − 3;3 − 5];**

[V,D] = eig(A)

```
V =
    0.7071   0.7071
    0.7071   0.7071

D =
    -2     0
     0    -2
```

Inspecting V, we see that there is only one linearly independent eigenvector, so A is not diagonalizable.

(d) **A = [1 0 0;0 1 1;0 1 1];**
[V,D] = eig(A)

```
V =
    1.0000        0         0
         0   0.7071    0.7071
         0   0.7071   -0.7071

D =
    1.0000        0         0
         0   2.0000         0
         0        0    0.0000
```

V′ ∗ V

```
ans =
    1.0000        0         0
         0   1.0000         0
         0        0    1.0000
```

Hence V is orthogonal. We should have expected this since A is symmetric.

Complex Numbers

Appendix B.1, p. A-11

2. (a) $-\frac{1}{5}+\frac{2}{5}i$. (b) $\frac{9}{10}-\frac{7}{10}i$. (c) $4-3i$. (d) $\frac{1}{26}-\frac{5}{26}i$.

4. $\sqrt{20}$. (b) $\sqrt{10}$. (c) $\sqrt{13}$. (d) $\sqrt{17}$.

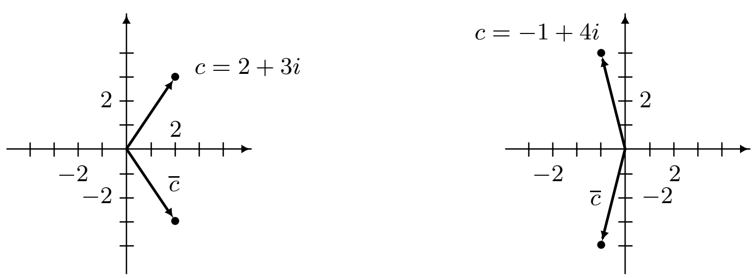
5. (a) $\mathrm{Re}(c_1+c_2)=\mathrm{Re}((a_1+a_2)+(b_1+b_2)i)=a_1+a_2=\mathrm{Re}(c_1)+\mathrm{Re}(c_2)$
   $\mathrm{Im}(c_1+c_2)=\mathrm{Im}((a_1+a_2)+(b_1+b_2)i)=b_1+b_2=\mathrm{Im}(c_1)+\mathrm{Im}(c_2)$

   (b) $\mathrm{Re}(kc)=\mathrm{Re}(ka+kbi)=ka=k\mathrm{Re}(c)$
   $\mathrm{Im}(kc)=\mathrm{Im}(ka+kbi)=kb=k\mathrm{Im}(c)$

   (c) No.

   (d) $\mathrm{Re}(c_1c_2)=\mathrm{Re}((a_1+b_1i)(a_2+b_2i))=\mathrm{Re}((a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i)=a_1a_2-b_1b_2\neq \mathrm{Re}(c_1)\mathrm{Re}(c_2)$

6.



8. (a) $\left(\overline{A+B}\right)_{ij}=\overline{a_{ij}+b_{ij}}=\overline{a_{ij}}+\overline{b_{ij}}=\left(\overline{A}\right)_{ij}+\left(\overline{B}\right)_{ij}$

   (b) $\left(\overline{kA}\right)_{ij}=\overline{ka_{ij}}=k\,\overline{a_{ij}}=k\left(\overline{A}\right)_{ij}$.

   (c) $\overline{CC^{-1}}=\overline{C^{-1}C}=\overline{I_n}=I_n$; thus $(\overline{C})^{-1}=\overline{C^{-1}}$.

10. (a) Hermitian, normal. (b) None. (c) Unitary, normal. (d) Normal.
    (e) Hermitian, normal. (f) None. (g) Normal. (h) Unitary, normal.
    (i) Unitary, normal. (j) Normal.

11. (a) $\overline{a_{ii}}=a_{ii}$, hence a_{ii} is real. (See Property 4 in Section B1.)

    (b) First, $\overline{A^T}=A$ implies that $A^T=\overline{A}$. Let $B=\dfrac{A+\overline{A}}{2}$. Then

$$\overline{B}=\overline{\left(\frac{A+\overline{A}}{2}\right)}=\frac{\overline{A}+\overline{\overline{A}}}{2}=\frac{\overline{A}+A}{2}=\frac{A+\overline{A}}{2}=B,$$

so B is a real matrix. Also,

$$B^T=\left(\frac{A+\overline{A}}{2}\right)^T=\frac{A^T+\overline{A}^T}{2}=\frac{A^T+\overline{A^T}}{2}=\frac{\overline{A}+A}{2}=\frac{A+\overline{A}}{2}=B$$

so B is symmetric.

Next, let $C = \dfrac{A - \overline{A}}{2i}$. Then

$$\overline{C} = \overline{\left(\frac{A - \overline{A}}{2i}\right)} = \frac{\overline{A} - \overline{\overline{A}}}{-2i} = \frac{A - \overline{A}}{2i} = C$$

so C is a real matrix. Also,

$$C^T = \left(\frac{A - \overline{A}}{2i}\right)^T = \frac{A^T - \overline{A}^T}{2i} = \frac{A^T - \overline{A^T}}{2i} = \frac{\overline{A} - A}{2i} = -\frac{A - \overline{A}}{2i} = -C$$

so C is also skew symmetric. Moreover, $A = B + iC$.

(c) If $A = A^T$ and $A = \overline{A}$, then $\overline{A^T} = \overline{A} = A$. Hence, A is Hermitian.

12. (a) If A is real and orthogonal, then $A^{-1} = A^T$ or $AA^T = I_n$. Hence A is unitary.

(b) $\overline{(A^T)^T}A^T = \overline{(A^T)}^T A^T = (A\overline{A^T})^T = I_n^T = I_n$. Note: $\overline{(A^T)^T} = (\overline{A^T})^T$.
Similarly, $A^T\overline{(A^T)^T} = I_n$.

(c) $\overline{(A^{-1})^T}A^{-1} = \overline{(A^T)^{-1}}A^{-1} = (\overline{A^T})^{-1}A^{-1} = (A\overline{A^T})^{-1} = I_n^{-1} = I_n$.
Note: $\overline{(A^{-1})^T} = \overline{(A^T)^{-1}}$ and $\overline{(A^T)^{-1}} = (\overline{A^T})^{-1}$. Similarly, $A^{-1}\overline{(A^{-1})^T} = I_n$.

13. (a) Let

$$B = \frac{A + \overline{A^T}}{2} \quad \text{and} \quad C = \frac{A - \overline{A^T}}{2i}.$$

Then

$$\overline{B^T} = \overline{\left(\frac{A + \overline{A^T}}{2}\right)^T} = \frac{\overline{A^T} + \overline{(\overline{A^T})^T}}{2} = \frac{\overline{A^T} + A}{2} = \frac{A + \overline{A^T}}{2} = B$$

so B is Hermitian. Also,

$$\overline{C^T} = \overline{\left(\frac{A - \overline{A^T}}{2i}\right)^T} = \frac{\overline{A^T} - \overline{(\overline{A^T})^T}}{-2i} = \frac{A - \overline{A^T}}{2i} = C$$

so C is Hermitian. Moreover, $A = B + iC$.

(b) We have

$$\begin{aligned}
\overline{A^T}A = \overline{(B^T + iC^T)}(B + iC) &= (\overline{B^T} + \overline{iC^T})(B + iC) \\
&= (B - iC)(B + iC) \\
&= B^2 - iCB + iBC - i^2C^2 \\
&= (B^2 + C^2) + i(BC - CB).
\end{aligned}$$

Similarly,

$$\begin{aligned}
A\overline{A^T} = (B + iC)\overline{(B^T + iC)^T} &= (B + iC)(\overline{B^T} + \overline{iC^T}) \\
&= (B + iC)(B - iC) \\
&= B^2 - iBC + iCB - i^2C^2 \\
&= (B^2 + C^2) + i(CB - BC).
\end{aligned}$$

Since $\overline{A^T}A = A\overline{A^T}$, we equate imaginary parts obtaining $BC - CB = CB - BC$, which implies that $BC = CB$. The steps are reversible, establishing the converse.

14. (a) If $\overline{A^T} = A$, then $\overline{A^T}A = A^2 = A\overline{A^T}$, so A is normal.
    (b) If $\overline{A^T} = A^{-1}$, then $\overline{A^T}A = A^{-1}A = AA^{-1} = A\overline{A^T}$, so A is normal.
    (c) One example is $\begin{bmatrix} i & i \\ i & i \end{bmatrix}$. Note that this matrix is not symmetric since it is not a real matrix.

15. Let $A = B + iC$ be skew Hermitian. Then $\overline{A^T} = -A$ so $B^T - iC^T = -B - iC$. Then $B^T = -B$ and $C^T = C$. Thus, B is skew symmetric and C is symmetric. Conversely, if B is skew symmetric and C is symmetric, then $B^T = -B$ and $C^T = C$ so $B^T - iC^T = -B - iC$ or $\overline{A^T} = -A$. Hence, A is skew Hermitian.

16. (a) $x = \dfrac{-1 \pm i\sqrt{3}}{2}$. (b) -2, $\pm i$. (c) 1, $\pm i$, -1, -1 (-1 is a double root).

18. (a) Possible answers: $A_1 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $A_2 = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$.

20. (a) Possible answers: $\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -i & 0 \\ 0 & 0 \end{bmatrix}$.
    (b) Possible answers: $\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$, $\begin{bmatrix} -i & i \\ i & -i \end{bmatrix}$.

Appendix B.2, p. A-20

2. (a) $x = -\frac{7}{30} - \frac{4}{30}i$, $y = -\frac{11}{15}(1 + 2i)$, $z = \frac{3}{5} - \frac{4}{5}i$.
   (b) $x = -1 + 4i$, $y = \frac{1}{2} + \frac{3}{2}i$, $z = 2 - i$.

4. (a) $4i$ (b) 0. (c) $-9 - 8i$. (d) -10.

6. (a) Yes. (b) No. (c) Yes.

7. (a) Let A and B be Hermitian and let k be a complex scalar. Then

$$(\overline{A + B})^T = (\overline{A} + \overline{B})^T = \overline{A}^T + \overline{B}^T = A + B,$$

so the sum of Hermitian matrices is again Hermitian. Next,

$$(\overline{kA})^T = \overline{kA}^T = \overline{k}A \neq kA,$$

so the set of Hermitian matrices is not closed under scalar multiplication and hence is not a complex subspace of C_{nn}.

   (b) From (a), we have closure of addition and since the scalars are real here, $\overline{k} = k$, hence $(\overline{kA})^T = kA$. Thus, W is a real subspace of the real vector space of $n \times n$ complex matrices.

8. The zero vector $\mathbf{0}$ is not unitary, so W cannot be a subspace.

10. (a) No. (b) No.

12. (a) $P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. (b) $P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$.
    (c) $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ i & 0 & -i \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ i & -i & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{bmatrix}$.

13. (a) Let A be Hermitian and suppose that $A\mathbf{x} = \lambda\mathbf{x}$, $\lambda \neq 0$. We show that $\lambda = \overline{\lambda}$. We have

$$(\overline{A\mathbf{x}})^T = \left(\overline{A}\,\overline{\mathbf{x}}\right)^T = \overline{\mathbf{x}}^T\,\overline{A} = \overline{\mathbf{x}}^T A.$$

Also, $(\overline{\lambda}\,\overline{\mathbf{x}})^T = \overline{\lambda}\,\overline{\mathbf{x}}^T$, so $\overline{\mathbf{x}}^T A = \overline{\lambda}\,\overline{\mathbf{x}}^T$. Multiplying both sides by $\mathbf{x}$ on the right, we obtain $\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\lambda}\overline{\mathbf{x}}^T\mathbf{x}$. However, $\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}^T\lambda\mathbf{x} = \lambda\overline{\mathbf{x}}^T\mathbf{x}$. Thus, $\overline{\lambda}\overline{\mathbf{x}}^T\mathbf{x} = \lambda\overline{\mathbf{x}}^T\mathbf{x}$. Then $(\lambda - \overline{\lambda})\overline{\mathbf{x}}^T\mathbf{x} = 0$ and since $\overline{\mathbf{x}^T}\mathbf{x} > 0$, we have $\lambda = \overline{\lambda}$.

(b) $\overline{A}^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & \overline{-i} \\ 0 & \overline{i} & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & i \\ 0 & -i & 2 \end{bmatrix} = A.$

(c) No, see 11(b). An eigenvector $\mathbf{x}$ associated with a real eigenvalue λ of a complex matrix A is in general complex, because $A\mathbf{x}$ is in general complex. Thus $\lambda\mathbf{x}$ must also be complex.

14. If A is unitary, then $\overline{A}^T = A^{-1}$. Let $A = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n\end{bmatrix}$. Since

$$I_n = A\overline{A^T} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n\end{bmatrix}\begin{bmatrix}\overline{\mathbf{u}_1^T} \\ \overline{\mathbf{u}_2^T} \\ \vdots \\ \overline{\mathbf{u}_n^T}\end{bmatrix},$$

then

$$\mathbf{u}_k\overline{\mathbf{u}_j^T} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

It follows that the columns $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ form an orthonormal set. The steps are reversible establishing the converse.

15. Let A be a skew symmetric matrix, so that $\overline{A^T} = -A$, and let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. We show that $\overline{\lambda} = -\lambda$. We have $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides of this equation by $\overline{\mathbf{x}^T}$ on the left we have $\overline{\mathbf{x}^T}A\mathbf{x} = \overline{\mathbf{x}^T}\lambda\mathbf{x}$. Taking the conjugate transpose of both sides yields

$$\overline{\mathbf{x}^T A^T\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}^T}\mathbf{x}.$$

Therefore $-\overline{\mathbf{x}^T}A\mathbf{x} = \overline{\lambda}\,\overline{\mathbf{x}^T}\mathbf{x}$, or $-\lambda\overline{\mathbf{x}^T}\,\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}^T}\mathbf{x}$, so $(\lambda + \overline{\lambda})(\overline{\mathbf{x}^T}\mathbf{x}) = 0$. Since $\mathbf{x} \neq \mathbf{0}$, $\overline{\mathbf{x}^T}\mathbf{x} \neq 0$, so $\overline{\lambda} = -\lambda$. Hence, the real part of λ is zero.