Interpolation

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- Given a function $y = f(x)$, we approximat $f(x)$ by a polynomial $P(x)$. We evaluate $f(x)$ at some points $x_0, x_1, x_2, \dots, x_n$

x	x_0	x_1	x_2	--	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	--	$f(x_n)$

I) Lagrange Interpolation Formula

$$P(x) = y_0\, \ell_0(x) + y_1\, \ell_1(x) + \dots + y_n\, \ell_n(x)$$

& $$\ell_0(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

$$\ell_1(x) = \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}$$

$$\vdots$$

$$\ell_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

<u>Ex</u>: Interpolate the given data, then find $f(2)$

$x = \quad 0 \quad 1 \quad 3 \quad 5$

$y = \quad -3 \quad 0 \quad 4 \quad 1$

<u>Solution</u>: We use Lagrange interpolation, we find agrange basis l_0, l_1, l_2, l_3

$$l_0 = \frac{(x-1)(x-3)(x-5)}{(-1)(-3)(-5)} = -\frac{1}{15}(x^3 - 9x^2 + 23x - 15)$$

$$l_1 = \frac{x(x-3)(x-5)}{1(-2)(-4)} = \frac{1}{8}(x^3 - 8x^2 + 15x)$$

$$l_2 = \frac{x(x-1)(x-5)}{3(2)(-2)} = -\frac{1}{12}(x^3 - 6x^2 + 5x)$$

$$l_3 = \frac{x(x-1)(x-3)}{5(4)(2)} = \frac{1}{40}(x^3 - 4x^2 + 3x)$$

$$P(x) = -3\,l_0 + 0\,l_1 + 4\,l_2 + 1\,l_3$$

$$= -\frac{13}{120}x^3 + \frac{1}{10}x^2 + \frac{361}{120}x - 3$$

$$\Rightarrow f(2) \simeq P(2) = 51/20$$

<u>ملحوظه</u>: درجه كثيرة الحدود $P(x)$ دائماً" $\geq n$

Example:

1) Use Lagrange's Formula to obtain a polynomial of least degree that fits the given data :-

(i) $x = 0 \quad 2 \quad 3$

$y = 7 \quad 11 \quad 28$

Solution:

The polynomial is of degree ≤ 2 &

$$P(x) = L_0(x) \cdot y_0 + L_1(x) \cdot y_1 + L_2(x)\, y_2$$

$$= 7L_0(x) + 11\, L_1(x) + 28\, L_2(x) \rightarrow (1)$$

where,
$$L_0(x) = \frac{(x-2)(x-3)}{(0-2)(0-3)} = \frac{1}{6}(x^2 - 5x + 6)$$

$$L_1(x) = \frac{(x-0)(x-3)}{(2-0)(2-3)} = -\frac{1}{2}(x^2 - 3x)$$

$$L_2(x) = \frac{(x-0)(x-2)}{(3-0)(3-2)} = \frac{1}{3}(x^2 - 2x)$$

Sub. in (1) $\Rightarrow P(x) = \frac{7}{6}(x^2 - 5x + 6) - \frac{11}{2}(x^2 - 3x) + \frac{28}{3}(x^2 - 2x$

$\Rightarrow P(x) = 5x^2 - 8x + 7$

(ii)

x :	0	2	3	4
y :	7	11	28	63

Solution: The polynomial is of degree ≤ 3 &

$$P(x) = L_0(x).y_0 + L_1(x).y_1 + L_2(x).y_2 + L_3(x).y$$

$$= 7L_0(x) + 11\,L_1(x) + 28L_2(x) + 63L_3(x) \quad \rightarrow (1)$$

where,

$$L_0(x) = \frac{(x-2)(x-3)(x-4)}{(0-2)(0-3)(0-4)} = -\frac{1}{24}(x-2)(x^2-7x+12)$$

$$= -\frac{1}{24}(x^3 - 9x^2 + 26x - 24)$$

$$L_1(x) = \frac{(x-0)(x-3)(x-4)}{(2-0)(2-3)(2-4)} = \frac{1}{4}x(x^2-7x+12)$$

$$= \frac{1}{4}(x^3 - 7x^2 + 12x)$$

$$L_2(x) = \frac{(x-0)(x-2)(x-4)}{(3-0)(3-2)(3-4)} = -\frac{1}{3}(x)(x^2-6x+8)$$

$$= -\frac{1}{3}(x^3 - 6x^2 + 8x)$$

$$L_3(x) = \frac{(x-0)(x-2)(x-3)}{(4-0)(4-2)(4-3)} = \frac{1}{8}x(x^2-5x+6)$$

$$= \frac{1}{8}(x^3 - 5x^2 + 6x)$$

$\Rightarrow$ Sub. in (1) we get $P(x) = x^3 - 2x + 7$

Example: Find a polynomial of least degree that fits the data:

$$\begin{array}{lcccc} x: & -1 & 4 & 1 & 0 \\ f(x): & -2 & 43 & 4 & -1 \end{array}$$

, then find the interpolated value at $x = 2$.

Solution: we arrange the values as,

$$\begin{array}{lcccc} x: & -1 & 0 & 1 & 4 \\ f(x): & -2 & -1 & 4 & 43 \end{array}$$

Using Lagrange Interpolation formula,

$$f(x) \simeq P(x) = -2L_0(x) - L_1(x) + 4L_2(x) + 43L_3(x)$$

where, $L_0(x) = \dfrac{(x-0)(x-1)(x-4)}{(-1-0)(-1-1)(-1-4)} = \dfrac{-1}{10}(x^3 - 5x^2 + 4x)$

$$L_1(x) = \frac{(x+1)(x-1)(x-4)}{(0+1)(0-1)(0-4)} = \frac{1}{4}(x^3 - 4x^2 - x + 4)$$

$$L_2(x) = \frac{(x+1)(x-0)(x-4)}{(1+1)(1-0)(1-4)} = -\frac{1}{6}(x^3 - 3x^2 - 4x)$$

$$L_3(x) = \frac{(x+1)(x-0)(x-1)}{(4+1)(4-0)(4-1)} = \frac{1}{60}(x^3 - x)$$

$$\Rightarrow f(x) \simeq P(x) = \frac{2}{10}(x^3 - 5x^2 + 4x) - \frac{1}{4}(x^3 - 4x^2 - x + 4)$$
$$- \frac{4}{6}(x^3 - 3x^2 - 4x) + \frac{43}{60}(x^3 - x) = 2x^2 + 3x - 1$$

$$f(2) \simeq P(2) = 2(4) + 3(2) - 1 = 13$$

Example:- Use Lagrange Formula with the data

X :	0.5	0.6	0.7.
$\cos x$:	0.87758	0.82534	0.76484

to find $\cos(0.54)$ by

(i) Linear interpolation

(ii) Quadratic interpolation.

Solution:- (i) Linear Interpolation : we approximal the curve by a linear fn. (Polyn. of degree 1)

$\Rightarrow$ We need 2 values only , we use

X :	0.5	0.6
$\cos x$:	0.87758	0.82534

$$\Rightarrow \cos x \simeq P(x) = y_0 L_0(x) + y_1 L_1(x)$$

$$= 0.87758\left(\frac{x-0.6}{0.5-0.6}\right) + 0.82534\left(\frac{x-0.5}{0.6-0.5}\right)$$

$$= -0.5224x + 1.13878$$

$$\Rightarrow \cos(0.54) \simeq -0.5224(0.54) + 1.13878$$

$$\simeq 0.856684$$

(ii) <u>Quadratic Interpolation</u> : we approximate the given data by a quadratic fn (Polyn. of degree 2) $\Rightarrow$ we use 3 values to find such Polynomial, we use

X:	0.5	0.6	0.7
$\cos X$:	0.87758	0.82534	0.76484

$$\cos X \simeq P(X) = 0.87758\, L_0(X) + 0.82534\, L_1(X) + 0.76484\, L_2(X)$$

$$L_0(X) = \frac{(X-0.6)(X-0.7)}{(0.5-0.6)(0.5-0.7)} = 50(X^2 - 1.3X + 0.42)$$

$$L_1(X) = \frac{(X-0.5)(X-0.7)}{(0.6-0.5)(0.6-0.7)} = -100(X^2 - 1.2X + 0.35)$$

$$L_2(X) = \frac{(X-0.5)(X-0.6)}{(0.7-0.5)(0.7-0.6)} = 50(X^2 - 1.1X + 0.3)$$

$$\cos X \simeq P(X) = 0.87758(50)(X^2 - 1.3X + 0.42)$$
$$+ 0.82534(-100)(X^2 - 1.2X + 0.35) + 0.76484(50)(X^2 - 1.1X + 0.$$

$$\Rightarrow \cos X \simeq -0.413X^2 - 0.0681X + 1.01488$$

$$\Rightarrow \cos(0.54) \simeq 0.85767$$

II) Divided Difference Interpolation

* If the given points are <u>not equally spaced</u>, we form the divided difference table

x_i	$f(x_i)$	1st divid. diff. $f[x_i, x_{i+1}]$	2nd divid. diff. $f[x_i, x_{i+1}, x_{i+2}]$
x_0	y_0		
		$f[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$	
x_1	y_1		$\dfrac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0}$
		$f[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$	
x_2	y_2		$\dfrac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1}$
		$f[x_2, x_3] = \dfrac{y_3 - y_2}{x_3 - x_2}$	
x_3	y_3		
$\vdots$		$\vdots$	$\vdots$

* Forward divided difference

$$f(x) \simeq P(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \cdots$$

* Backward divided difference

$$f(x) \simeq P(x) = f(x_n) + (x - x_n) f[x_{n-1}, x_n] + (x - x_n)(x - x_{n-1}) f[x_{n-2}, x_{n-1}, x_n] + \cdots$$

Example:- Find the divided difference table for the data,

$x:$	-2	-1	0	3	7
$f(x):$	71	21	1	121	28925

Solution: Points are not equally spaced, we form the divided difference table as follow :-

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	--	--
-2	71				
		$\frac{21-71}{-1-(-2)} = -50$			
-1	21		$\frac{-20-(-50)}{0-(-2)} = 15$		
		$\frac{1-21}{0-(-1)} = -20$		$\frac{15-15}{3-(-2)} = 0$	
0	1		$\frac{40-(-20)}{3-(-1)} = 15$		$\frac{126-0}{7-(-2)} = 14$
		$\frac{121-1}{3-0} = 40$		$\frac{1023-15}{7-(-1)} = 126$	
3	121		$\frac{7201-40}{7-0} = 1023$		
		$\frac{28925-121}{7-3} = 7201$			
7	28925				

We will show now how to use the divided difference table to interpolate these points.

- Forward Interpolation

$$f(x) \simeq P(x) = 71 + (x+2)(-50) + (x+2)(x+1)(15)$$
$$+ (x+2)(x+1)(x-0)(0) +$$
$$(x+2)(x+1)(x)(x-3)(14)$$
$$= 14x^4 - 83x^2 - 89x + 1$$

- Backward Interpolation

$$f(x) \simeq P(x) = 28925 + (x-7)(7201) +$$
$$(x-7)(x-3)(1023) + (x-7)(x-3)(x-0)(126)$$
$$+ (x-7)(x-3)(x-0)(x+1)(14)$$
$$= 14x^4 - 83x^2 - 89x + 1$$

* ملحوظة: لايجاد قيمة تقريبية لـ $f(a)$:

1) اذا كانت a في النصف العلوي من الجدول ، نختار x_0 بالقيمة الاقل منها و نستخدم Forward Interpolation

2) اذا كانت a في النصف الاسفل من الجدول ، نختار x_n بالقيمة الاكبر منها و نستخدم Backward Interpolation

...rward & Backward Newton Formulas for Equally Spaced Points :-

The interpolating Polynomial $P(x)$ for an equally spaced points can be obtained from the difference table from the formula :-

1) Forward Newton Interpolation Formula,

let $h =$ step size & $s = \frac{x - x_0}{h}$

$$f(x) \simeq P(x) = f(x_0) + s\,\Delta f + \frac{s(s-1)}{2!}\Delta^2 f + \frac{s(s-1)(s-2)}{3!}\Delta^3 f + \cdots$$

2) Backward Newton Interpolation Formula,

- We denote the differences by ∇f instead of Δf

let $h =$ step size & $s = \frac{x - x_0}{h}$

$$f(x) \simeq P(x) = f(x_0) + s\,\nabla f + \frac{s(s+1)}{2!}\nabla^2 f + \frac{s(s+1)(s+2)}{3!}\nabla^3 f + \cdots$$

Example: Find the difference table for the data

x:	-3	-1	1	3	5
$f(x)$:	252	-2	0	66	772

Solution: Points are equally spaced, we form the difference table as follow :-

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
-3	252				
		$-2-252=-254$			
-1	-2		$2-(-254)=256$		
		$0-(-2)=2$		$64-256=-192$	
1	0		$66-2=64$		$576-(-192)=76$
		$66-0=66$		$640-64=576$	
3	66		$706-66=640$		
		$772-66=706$			
5	772				

We will show now how to use the difference table to interpolate these points.

* For example, for the previous example & its difference table ;

- Forward Interpolation with $x_0 = -3$

we have $h = 2$, $s = \frac{x-(-3)}{2} = \frac{x+3}{2}$

$$f(x) \simeq P(x) = f(x_0) + s\Delta f + \frac{s(s-1)}{2!}\Delta^2 f + \frac{s(s-1)(s-2)}{3!}\Delta^3 f + \cdots$$

$$= 252 + \left(\frac{x+3}{2}\right)(-254) + \frac{1}{2!}\left(\frac{x+3}{2}\right)\left(\frac{x+3}{2}-1\right)(256)$$

$$+ \frac{1}{3!}\left(\frac{x+3}{2}\right)\left(\frac{x+3}{2}-1\right)\left(\frac{x+3}{2}-2\right)(-192)$$

$$+ \frac{1}{4!}\left(\frac{x+3}{2}\right)\left(\frac{x+3}{2}-1\right)\left(\frac{x+3}{2}-2\right)\left(\frac{x+3}{2}-3\right)(768)$$

$$= 252 - 127(x+3) + 32(x+3)(x+1) - 4(x+3)(x+1)(x-1)$$

$$+ 2(x+3)(x+1)(x-1)(x-3)$$

$$= 252 - 127(x+3) + 32(x^2+4x+3) - 4(x^3+3x^2-x-3)$$

$$+ 2(x^4-10x^2+9) \quad = 2x^4 - 4x^3 + 5x - 3$$

- Backward Interpolation with $x_0 = 5$ gives ,

We have $h = 2$, $s = \frac{x-5}{2}$

$$f(x) \simeq P(x) = 772 + \left(\frac{x-5}{2}\right)(706) + \left(\frac{x-5}{2}\right)\left(\frac{x-5}{2}+1\right)\cdot\frac{1}{2!}(64$$

$$+\frac{1}{3!}\left(\frac{x-5}{2}\right)\left(\frac{x-5}{2}+1\right)\left(\frac{x-5}{2}+2\right)(576)$$

$$+\frac{1}{4!}\left(\frac{x-5}{2}\right)\left(\frac{x-5}{2}+1\right)\left(\frac{x-5}{2}+2\right)\left(\frac{x-5}{2}+3\right)(768$$

ثم نفك الاقواس مثل الحالة السابقة .

- <u>To approximate the value of $f(-2)$</u>

نختار x_0 بحيث تكون أقرب نقطة من النقط المعطاة للنقطة -2

$\Rightarrow$ let $x_0 = -3$ & use Forward interpolation formula

أكثر دقة

$$\Rightarrow S = \frac{-2-(-3)}{2} = \frac{1}{2}$$

$$f(-2) \simeq P(-2) = 252 + \frac{1}{2}(-254) + \frac{(\frac{1}{2})(\frac{-1}{2})}{2!}(256)$$

$$+ \frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})}{3!}(-192) + \frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})}{4!}(768) = 51$$

- <u>To approximate the value of $f(2)$</u>

نختار x_0 نقطة من النقط المعطاة والاقرب للنقطة المطلوبة (2)

$\Rightarrow$ let $x_0 = 3$ & use Backward interpolation formula

أكثر دقة

$$\Rightarrow S = \frac{2-3}{2} = -\frac{1}{2}$$

$$f(2) \simeq P(2) = 66 + (\tfrac{-1}{2})(66) + \frac{1}{2!}(\tfrac{-1}{2})(\tfrac{1}{2})(66)$$

$$+\frac{1}{3!}(\tfrac{-1}{2})(\tfrac{1}{2})(\tfrac{3}{2})(64) + \frac{1}{4!}(\tfrac{-1}{2})(\tfrac{1}{2})(\tfrac{3}{2})(\tfrac{5}{2})(-192) = 28.2$$

Example: Use the divided difference method to obtain a polynomial of least degree that fits the data

x:	1	3	-2	4	5
y:	2	6	-1	-4	2

Solution:-

It is better to arrange the data as

x:	-2	1	3	4	5
y:	-1	2	6	-4	2

the points are not equally spaced => divided diff.

x_i	$f(x_i)$	$f(x_i, x_{i+1})$	$f(x_i, x_{i+1}, x_{i+2})$	..	..
-2	$\underline{-1}$				
		$\underline{1}$			
1	2		$\underline{1/5}$		
		2		$\underline{-7/10}$	
3	6		-4		$\underline{37/70}$
		-10		3	
4	-4		8		
		6			
5	2				

Using forward interpolation formula:

$$P(x) = -1 + 1(x+2) + \frac{1}{5}(x+2)(x-1) - \frac{7}{10}(x+2)(x-1)(x-3)$$
$$+ \frac{37}{70}(x+2)(x-1)(x-3)(x-4)$$

$$= -1 + (x+2) + \frac{1}{5}(x^2+x-2) - \frac{7}{10}(x^3-2x^2-5x+6)$$
$$+ \frac{37}{70}(x^4-6x^3+3x^2+26x-24)$$

$$\Rightarrow P(x) = \frac{37}{70}x^4 - \frac{271}{70}x^3 + \frac{223}{70}x^2 + \frac{1291}{70}x - \frac{114}{7}$$

Note that: Backward Interpolation can be used also, as

$$P(x) = 2 + 6(x-5) + 8(x-5)(x-4)$$
$$+ 3(x-5)(x-4)(x-3) + \frac{37}{70}(x-5)(x-4)(x-3)(x-1)$$

$$= 2 + 6(x-5) + 8(x^2-9x+20) + 3(x^3-12x^2+47x-60)$$
$$+ \frac{37}{70}(x^4-13x^3+59x^2-107x+60)$$

$$= \frac{37}{70}x^4 - \frac{271}{70}x^3 + \frac{223}{70}x^2 + \frac{1291}{70}x - \frac{114}{7}$$

Example: Use the divided difference method to find the polynomial that best fits the data

x:	3	4	6	10	15
$f(x)$:	45	116	414	1970	6705

, Hence approxim[ate] the value of $f(4.4)$

Solution: Points are not equally spaced, the divided difference table is as follow,

x_i	$f[x_i]$	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	--	--
3	45				
		71			
4	116		26		
		149		2	
6	414		40		
		389		2	
10	1970		62		
		947			
15	6705				

$$f(x) \simeq P(x) = 45 + 71(x-3) + 26(x-3)(x-4)$$
$$+ 2(x-3)(x-4)(x-6) \quad = 2x^3 - 3x$$

* To approximate $f(4.4)$ we use forward divided difference with $x_0 = 4$,

$$f(4.4) \simeq 116 + 149(4.4-4) + 40(4.4-4)(4.4-6)$$
$$+ 2(4.4-4)(4.4-6)(4.4-10) \quad = 157.168$$

Example: Find the Polynomial of degree ≤ 2 that passes through the three points

$(0, 1)$, $(-1, 2)$ & $(1, 3)$

Solution:-

We use Lagrange Interpolation Formula

x:	-1	0	1
y:	2	1	3

$$\Rightarrow P(x) = 2L_0(x) + 1.L_1(x) + 3L_2(x) \longrightarrow (1)$$

where,

$$L_0(x) = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}(x^2 - x)$$

$$L_1(x) = \frac{(x+1)(x-1)}{(0+1)(0-1)} = -(x^2 - 1)$$

$$L_2(x) = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{1}{2}(x^2 + x)$$

Subsitute in (1),

$$P(x) = 2 \cdot \frac{1}{2}(x^2 - x) - (x^2 - 1) + 3 \cdot \frac{1}{2}(x^2 + x)$$

$$\Rightarrow P(x) = \frac{3}{2}x^2 + \frac{1}{2}x + 1$$

Example:- Given the data

X:	0	1	2	3	4
y:	1	-3	25	129	381

Find the best approximation for $f(\frac{1}{2})$, $f(\frac{6}{5})$, $f(\frac{18}{5})$

Solution:- The Points are Equally spaced , $h=1$

X	y	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	1				
		-4			
1	-3		32		
		28		44	
2	25		76		28
		104		72	
3	129		148		
		252			
4	381				

* $f(\frac{1}{2})$: use $x_0=0$ & forward interpolation formula,

$$S=\frac{1/2-0}{1}=1/2$$

$$f(\tfrac{1}{2})\cong 1+\tfrac{1}{2}(-4)+\frac{(1/2)(-1/2)}{2!}(32)+\frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})}{3!}(44)$$

$$+\frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})}{4!}(28)=-\frac{107}{32}$$

* $f(\frac{6}{5})$: use $x_0 = 1$ & forward interpolation formula.

$$S = \frac{6/5 - 1}{1} = 1/5$$

$$f(\tfrac{6}{5}) \cong -3 + \tfrac{1}{5}(28) + \frac{(\frac{1}{5})(-\frac{4}{5})}{2!}(76) + \frac{(\frac{1}{5})(-\frac{4}{5})(-\frac{9}{5})}{3!}(72$$

$$= -\frac{3}{125}$$

* For $f(\frac{18}{5})$: use $x_0 = 4$ & backward interpolation formula.

$$S = \frac{18/5 - 4}{1} = -2/5$$

$$f(\tfrac{18}{5}) \cong 381 + (-\tfrac{2}{5})(252) + \frac{(-\frac{2}{5})(\frac{3}{5})}{2!}(148)$$

$$+ \frac{(-\frac{2}{5})(\frac{3}{5})(\frac{8}{5})}{3!}(72) + \frac{(-\frac{2}{5})(\frac{3}{5})(\frac{8}{5})(\frac{13}{5})}{4!}(28)$$

$$= 256.6672$$

Example: By 2 methods, interpolate the given data

x:	1	3	-1	5
y:	-3	39	-21	345

Hence, approximate the value of y at $x = \frac{11}{2}$